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Slice regular functions on real alternative algebras

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Contents

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Setting

Let $(A, {}^c)$ be a finite dimensional, real, alternative -algebra, with unity 1_A.
Identify $\mathbb{R}$ with $\text{span}_{\mathbb{R}}\{1_A\}$ and let $Im(A) := \{x \in A \mid x^2 \in \mathbb{R}, x \notin \mathbb{R}\} \cup \{0\}$.

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Recall the properties of the antiinvolution $A \ni x \mapsto x^c \in A$:

$$(x^c)^c = x$$

$$(xy)^c = y^c x^c$$

$$x \in \mathbb{R} \implies x^c = x.$$

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$$x \in \mathbb{R} \implies x^c = x.$$

Define two maps

the trace

$$t(x) := x + x^c;$$

the norm

$$n(x) := xx^c.$$

They determine three subsets of A:

the normal cone

$$N_A := \{x \in A \mid n(x) = n(x^c) \in \mathbb{R} \setminus \{0\}\} \cup \{0\};$$

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$$Q_A := \{x \in A \mid t(x), n(x) \in \mathbb{R},\ 4n(x) > t(x)^2\} \cup \mathbb{R};$$

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$$Q_A := \{x \in A \mid t(x), n(x) \in \mathbb{R},\ 4n(x) > t(x)^2\} \cup \mathbb{R};$$

the sphere of $\sqrt{-1}$

$$\mathbb{S}_A := \{J \in Q_A \mid J^2 = -1\} \subset Q_A.$$

Finally, $\forall J \in \mathbb{S}_A$, define $\mathbb{C}_J := \langle 1_A, J \rangle \cong \mathbb{C}$.

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Examples

1. If $(A, {}^c) = (\mathbb{H}, \ )$ with the usual conjugation, $t(x), n(x) \in \mathbb{R}$, so $N_{(\mathbb{H}, \ )} = Q_{(\mathbb{H}, \ )} = \mathbb{H}$.

Examples

1. If $(A, {}^c) = (\mathbb{H}, \;)$ with the usual conjugation, $t(x), n(x) \; 2 \; \mathbb{R}$, so $N_{(\mathbb{H}, \;)} = Q_{(\mathbb{H}, \;)} = \mathbb{H}$.
2. For $q = q_0 + q_1 i + q_2 j + q_3 k \; 2 \; \mathbb{H}$ define the antiinvolution

$$q \;\; = q_0 + q_1 i + q_2 j \quad q_3 k.$$

We see that

$$Q_{(\mathbb{H}, \;)} = \mathbb{R} \quad \mathbb{R}k \; ' \; \mathbb{C} \quad \text{and} \quad \mathbb{S}_{(\mathbb{H}, \;)} = f \quad k g.$$

Properties of the cones

Proposition

1. *N_A and Q_A are cones of A and $Q_A \quad N_A$;*
2. *each $x\ 2\ N_A\ n\ f0g$ is invertible, with*

$$x^{\ \ 1} = \frac{x^c}{n(x)};$$

3. $S_A = fJ\ 2\ A\ j\ t(J) = 0, n(J) = 1g;$
4. $Q_A = A \quad (\ ) \quad A = \mathrm{C}, \mathrm{H}, \mathrm{O}.$

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The relevance of the quadratic cone Q_A comes from the following

Proposition

Every $x \in Q_A$ can be decomposed uniquely in

$$x = Re(x) + Im(x),$$

where $Re(x) \in \mathbb{R}$ and $Im(x) \in Im(A) \cap Q_A$, with $t(Im(x)) = 0$. Moreover,

$$Q_A = \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$$

Proof

Set

$$Re(x) := \frac{t(x)}{2} = \frac{x + x^c}{2} \qquad Im(x) := y := x \quad Re(x) = \frac{x \quad x^c}{2}.$$

Proof

Set

$$Re(x) := \frac{t(x)}{2} = \frac{x + x^c}{2} \qquad Im(x) := y := x \quad Re(x) = \frac{x \quad x^c}{2}.$$

$x\ 2\ Q_A \ =) \ Re(x)\ 2\ \mathrm{R}, t(y) = \frac{x \quad x^c}{2} + \frac{x^c \quad x}{2} = 0, \ =) \ y^c = \quad y,$
so $y^2 = \quad yy^c = \quad n(y)\ 2\ \mathrm{R}$, but

$$4y^2 = t(x)^2 \quad 4n(x) < 0, \qquad (x\ 2\ Q_A)$$

so $y\ 2\ Im(A)$.

Finally, set $J := \frac{y}{\sqrt{n(y)}} \implies x = Re(x) + J\sqrt{n(y)} \in \mathbb{C}_J$, with

$$t(J) = 0, \qquad n(J) = 1,$$

so $J \in \mathbb{S}_A$.

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Stem functions

Let $A_{\mathbb{C}} := A \quad \mathbb{C} = A \quad iA$. We can consider two antiinvolution over $A_{\mathbb{C}}$:

- the complex conjugation $\quad \overline{x + iy} := x \quad iy$;
- the antiinvolution from A $\quad (x + iy)^c := x^c + iy^c$.

Stem functions

Let $A_{\mathbb{C}} := A \otimes \mathbb{C} = A \oplus iA$. We can consider two antiinvolution over $A_{\mathbb{C}}$:

- the complex conjugation $\overline{x + iy} := x - iy$;
- the antiinvolution from A $(x + iy)^c := x^c + iy^c$.

Definition

A Stem function $F : D \to A_{\mathbb{C}}$ is a complex intrinsic function, i.e.

$$F(\bar{z}) = \overline{F(z)} \quad \forall z \in D \mid \bar{z} \in D.$$

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Definition

A Stem function $F : D \to A_{\mathbb{C}}$ is a complex intrinsic function, i.e.

$$F(z) = \overline{F(z)} \qquad \forall z \in D \mid z \in D.$$

If $F(z) = F_1(z) + iF_2(z)$,

$$F_1(z) = F_1(z) \qquad \text{and} \qquad F_2(z) = -F_2(z).$$

Slice functions

For every $J \in \mathbb{S}_A$, define two functions

$$\varphi_J : \mathbb{C} \to \mathbb{C}_J \subseteq A \qquad\qquad \Phi_J : A_{\mathbb{C}} \to A$$
$$\alpha + i\beta \mapsto \alpha + J\beta \qquad\qquad x + iy \mapsto x + Jy$$

and for $D \subseteq \mathbb{C}$, let $\Omega_D := \bigcup_{J \in \mathbb{S}_A} \varphi_J(D) = \{\alpha + J\beta \mid \alpha + i\beta \in D, J \in \mathbb{S}_A\}$.

Slice functions

For every $J \in S_A$, de ne two functions

$$\varphi_J : \mathbb{C} \xrightarrow{\varphi} C_J \quad A \qquad\qquad {}_J : A_{\mathbb{C}} \to \quad A$$

$$+ i \quad \mapsto \quad + J \qquad\qquad x + iy \mapsto x + Jy$$

and for $D \quad \mathbb{C}$, let ${}_D := \bigcup_{J \in S_A} \varphi_J(D) = \{ \quad + J \mid \quad + i \in D; J \in S_A \}$.

De nition

Any Stem function $F : D \to A_{\mathbb{C}}$ induces a slice function $f := I(F) :$ ${}_D \quad A \to A$, de ned on every $D_J := {}_J \cap C_J$ as

$$f := {}_J \quad F \quad \varphi_J^{-1}.$$

The set of Slice functions over ${}_D$ is denoted by $S({}_D)$.

This means that if $x = \quad + J \quad 2\ C_J$ and $F = F_1 + iF_2$, then

$$f(x) = f(\quad + J \quad) = F_1(\quad + i \quad) + JF_2(\quad + i \quad):$$

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The de nition is well posed:

if $x\ 2\ R = \bigcap^{T}_{J 2 S_A} C_J, \qquad f(x) = F_1(x) \qquad 8J$

$x = \quad + J \quad = \quad + (\quad J)(\quad) \ 2\ C_J \backslash\ C_{\ J}$, but

$$f(\quad + (\quad J)(\quad)) = \quad {}_J(F\overline{(\quad + i \quad)}) = \quad {}_J(F_1(\quad + i \quad) +$$
$$iF_2(\quad + i \quad)) = F_1(\quad + i \quad) + JF_2(\quad + i \quad) = f(\quad + J \quad):$$

Example

If $A = \mathbb{H}$, every polynomial with right coefficients

$$P(x) = \sum_{j=0}^{n} x^j a_j, \qquad a_j \in \mathbb{H}$$

is a stem function, induced by

$$F(z) = \sum_{j=0}^{n} z^j a_j = \sum_{j=0}^{n} \left[\mathrm{Re}(z^j) a_j + i(\mathrm{Im}(z^j) a_j) \right].$$

Slice regular functions on H

We recall that a function f : ! H is slice regular if for every J 2 S the function $f_J := f j_{\setminus C_J}$ is holomorphic with respect to the holomorphic structure induced by the (natural) identi cation of C_J with C.
In addition, is a S-domain if \ R 6= ; and $_J :=$ [C_J is connected for every J 2 S. A subset T of H is symmetric if

$$x + Jy \ 2 \ T \ ) \ x + yS \quad T:$$

Theorem (Identity Principle + Representation Formula)

Let be a slice domain, J 2 S and f : ! H be slice regular, then f is uniquely determined by f_J.
If in addition is symmetric, then there are unique functions b(x;y) and c(x;y) so that f(x + yJ) = b(x;y) + c(x;y)J.

General case

Let J 2 S_A, denote the upper half plane by

$$C_J^+ := f\,x = \quad + \quad J\ 2\ A\,j \qquad 0g$$

Theorem (Identity Principle)

Let $J;K$ 2 S_A with $J \quad K$ invertible. Then every slice function f 2 $S(\ _D)$ is uniquely determined by its value on C_J^+ [C_K^+. In particular, when $K = \quad J$, we get that f is completely determined by $f_J := f\,j_{\ _D \backslash C_J}$.

Proof. Let $f_J^+ := f\,j_{\ _D \backslash C_J^+}$, it is suf cient to deduce the stem function $F : C\ ! \quad A_C$ such that $I\ (F) = f$ from f_J^+ and f_K^+. Observe that:

$$f_J^+(\quad + \quad J) \quad f_K^+(\quad + \quad K) = (J \quad K)F_2(\quad + i \quad)$$

which holds for every $+ i \; 2 \; D$ with $\; 0$. In particular, when $\; = 0$

$$0 = f_J^+(\;) \quad f_K^+(\;) = (J \quad K)F_2(\;);$$

hence $F_2 \quad 0$ on $C_J \setminus C_K$. Thus F_2 is determined on the whole D by oddness. Finally,

$$f_J^+(\; + \; J) \quad JF_2(\; + i \;) = F_1(\; + i \;)$$

de nes an even function on D such that $F = F_1 + iF_2$, as desired.

Proposition (Representation Formula)

Let $f \; 2 \; S(\;_D)$ and let $J \; 2 \; S_A$. Then

$$f(\; + \; I) = \frac{1}{2}(f(\; + \; J) + f(\qquad J)) + \frac{I}{2}(f(\; + \; J) \quad f(\qquad J))$$

for every $\; + \; I \; 2 \; D_I := \;_D \setminus C_I$.

De nition

Let f 2 $S(\quad_D)$. We call the spherical value of f at x 2 $\quad_D$ the element of A

$$_S f(x) := \frac{1}{2}(f(x) + f(x^c)) :$$

We call the spherical derivative of f at x 2 $\quad_D$ n R the element of A

$$@_S f(x) := \frac{1}{2}\mathrm{Im}(x)^{\quad 1}(f(x) \quad f(x^c)) :$$

Both $_S f$ and $@_S f$ are slice functions of their domain, indeed they are induced by the stem functions $F_1(z)$ on D and $\mathrm{Im}(z)^{\quad 1}F_2(z)$ on D n R. Since these stem functions are A-valued, $_S$ and $@_S$ are constant on

$$S_{x=\quad+\quad J} := f\; y\; 2\; Q_A\; j\; y = \quad + \quad I; I\; 2\; S_A g:$$

Remark

1. $\partial_s(\partial_s f)$ 0 and $\partial_s(\ _S f)$ 0;
2. $\partial_s f(x) =$ 0 if and only if f is constant on S_x. In this case $f(x) = \ _S(x)$.
3. It is suf cient to assume F_2 of class C^1 to extend continuously $\partial_s f$ to the whole $\ _D$.

Then, for every x 2 $\ _D$ it holds:

$$f(x) = \ _S(x) + \mathrm{Im}(x)\partial_s f(x):$$

Smoothness of slice functions

Proposition

Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ be a slice function. Then:

1. If $F \in \mathcal{C}^0(D)$, then $f \in \mathcal{C}^0(\Omega_D)$. Moreover, $f_s^\circ \in \mathcal{C}^0(\Omega_D)$ and $\partial_s f \in \mathcal{C}^0(\Omega_D \setminus \mathbb{R})$.
2. If $F \in \mathcal{C}^{2s+1}(D)$ for some $s \in \mathbb{N}$, then $f, f_s^\circ, \partial_s f \in \mathcal{C}^s(\Omega_D)$. In particular, if $F \in \mathcal{C}^\infty(D)$, then $f, f_s^\circ, \partial_s f \in \mathcal{C}^\infty(\Omega_D)$.
3. If $F \in \mathcal{C}^\omega(D)$, then $f, f_s^\circ, \partial_s f \in \mathcal{C}^\omega(\Omega_D)$.

Denote by

$$\mathcal{S}^\tau(\Omega_D) := \{ f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D) \mid F \in \mathcal{C}^\tau(D) \},$$

where $\tau = s \in \mathbb{N}, \infty, \omega$.

If $f = \mathcal{I}(F) \in S^1(\Omega_D)$ and $z = \alpha + i\beta \in D$, define the A-stem functions

$$\frac{\partial F}{\partial z} := \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} - i\frac{\partial F}{\partial \beta}\right) \quad \text{and} \quad \frac{\partial F}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial F}{\partial \alpha} + i\frac{\partial F}{\partial \beta}\right)$$

Definition

If $f = \mathcal{I}(F) \in S^1(\Omega_D)$. We set the continous slice regular functions

$$\frac{\partial f}{\partial x} := \mathcal{I}\left(\frac{\partial F}{\partial z}\right) \quad \text{and} \quad \frac{\partial f}{\partial x^c} := \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right)$$

Slice regular functions

Consider the (natural) complex structure over $A_{\mathbb{C}}$. A stem function F is holomorphic if F_1 and F_2 satisfy the Cauchy Riemann relations:

$$\frac{\partial F}{\partial \bar{z}} \equiv 0.$$

Definition

A slice function $f \in \mathcal{S}^1(\Omega_D)$ is slice regular if its stem function F such that $f = \mathcal{I}(F)$ is holomorphic. The set of slice regular functions is denoted by $\mathcal{SR}(\Omega_D)$.

Remark. A slice function $f \in \mathcal{SR}(\Omega_D)$ if and only if $\frac{\partial f}{\partial x^c} \equiv 0$.

Proposition

Let f 2 S^1($_D$). The following are equivalent:

1. f is slice regular;
2. f_J is holomorphic for every J 2 S_A.

Proof. Recall that f_J(+ J) = F_1(+ i) + JF_2(+ i). Then

()) if F is holomorphic, it holds:

$$\frac{@f_J}{@} + J\frac{@f_J}{@} = \frac{@F_1}{@} \quad \frac{@F_2}{@} + J \quad \frac{@F_2}{@} + \frac{@F_1}{@} = 0:$$

(( ) if f_J is holomorphic for every J 2 S_A, it holds:

$$0 = \frac{@f_J}{@} + J\frac{@f_J}{@} = \frac{@F_1}{@} \quad \frac{@F_2}{@} + J \quad \frac{@F_2}{@} + \frac{@F_1}{@} \quad :$$

Product of slice functions

If F and G are stem functions, then $F \quad G$ is stem as well. Indeed:

$$\begin{aligned} FG(\overline{z}) &= F_1(\overline{z})G_1(\overline{z}) \quad F_2(\overline{z})G_2(\overline{z}) + i(F_1(\overline{z})G_2(\overline{z}) + F_2(\overline{z})G_1(\overline{z})) \\ &= F_1(z)G_1(z) \quad F_2(z)G_2(z) + i(\quad F_1(z)G_2(z) \quad F_2(z)G_1(z)) \\ &= \overline{FG}(z). \end{aligned}$$

Definition

Let $f = I(F)$ and $g = I(G)$ be slice functions, then the product of f and g is the slice function

$$f \quad g := I(FG).$$

Proposition

If $f, g \in SR(\Omega_D)$, then $f \cdot g \in SR(\Omega_D)$ as well.

Proposition

If $f, g \in \mathcal{SR}(\Omega_D)$, then $f \cdot g \in \mathcal{SR}(\Omega_D)$ as well.

Definition

A slice function f is said to be real if the A-valued components F_1, F_2 of its stem function are real-valued.

Proposition

$f \in \mathcal{S}(\Omega_D)$ is real if and only if $f(\mathbb{C}_J \cap \Omega_D) \subset \mathbb{C}_J$ for every $J \in \mathbb{S}_A$.

Proof. Assume $f(x) = F_1(z) + JF_2(z) \in \mathbb{C}_J$ for every $x = \alpha + J\beta \in \Omega_D$. Then $f(x^c) \in \mathbb{C}_J$ as well, that is $\overline{f(x^c)} = f(\alpha - J\beta) = F_1(z) - JF_2(z) \in \mathbb{C}_J$. This means that $F_1(z), F_2(z) \in \bigcap_J \mathbb{C}_J = \mathbb{R}$.

Normal function

De nition

Let $f = I(F)\ 2\ S(\ _D)$. Set $F^c(z) := F_1(z)^c + iF_2(z)^c$ and de ne

1. $f^c := I(F^c)$;
2. $CN(F) := FF^c = F_1F_1^c \quad F_2F_2^c + i(F_1F_2^c + F_2F_1^c) = n(F_1) \quad n(F_2) + it(F_1F_2^c)$;
3. $N(f) := I(CN(F))$, called the normal function of f.

If $A = H$, the norm and the trace are real valued, so it is $N(f)$. This is not true in a general algebra. Thus we have the following

Admissible functions

Definition

A Slice function $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ is called admissible if

$$\langle F_1(z), F_2(z) \rangle \subseteq N_A \qquad \forall z \in D.$$

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Proposition

f admissible $\Longrightarrow$ $N(f)$ real.

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Proposition

f admissible $\Rightarrow$ $N(f)$ real.

Proof. $N(f) = \mathcal{I}(G)$, with $G_1 := n(F_1) \quad n(F_2)$ and $G_2 := t(F_1 F_2^c)$. f admissible $\Rightarrow$ $n(F_1), n(F_2) \in \mathbb{R}$, so $G_1 \in \mathbb{R}$. But also

$$G_2 = n(F_1 + F_2) \quad n(F_1) \quad n(F_2) \in \mathbb{R}.$$

The converse may not hold. Take, in R_3,

$$p(x) := x^2 e_{123} + x(e_1 + e_{23}) + 1 = I(P);$$

with $P_2(\quad + i \quad) = 2 \quad e_{123} + \quad (e_1 + e_{23})$.
It is not admissible, since

$$n(P_2(i)) = (e_1 + e_{23})(e_1 + e_{23})^c = 2(1 \quad e_{123}) \notin R$$

even if

$$N(p) = (x^2 + 1)^2$$

is real.

Zeros

For an admissible slice function $f \in \mathcal{S}(\Omega_D)$, consider its zero set $V(f) := \{x \in Q_A \mid f(x) = 0\}$. The structure of $V(f)$ is shown in the following

Theorem

Let $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ be an admissible function, $x = \alpha + J\beta \in \Omega_D \cap \mathbb{C}_J$ and $z = \alpha + i\beta$. Then, it can happen:

1. $\mathbb{S}_x \cap V(f) = \emptyset \iff CN(F)(z) \neq 0$;
2. $\mathbb{S}_x \subseteq V(f) \iff F(z) = 0$;
3. $\#\{\mathbb{S}_x \cap V(f)\} = 1 \iff F(z) \neq 0$, but $CN(F)(z) = 0$.

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1. $S_x \cap V(f) = \emptyset \iff CN(F)(z) \neq 0$;
2. $S_x \subseteq V(f) \iff F(z) = 0$;
3. $\#\{S_x \cap V(f)\} = 1 \iff F(z) \neq 0$, but $CN(F)(z) = 0$.

If (2) holds, x is called spherical zero, if (3) holds, x is called isolated zero (Camshaft effect).

Lemma

Let $w = x + iy \in A_{\mathbb{C}}$, with $y \in N_A \setminus \{0\}$. Let $cn(w) := ww^c$. Then

1. $cn(w) = 0 \iff w = 0$ or $\exists!\, K \in S_A \mid x + Ky = 0$;
2. If $\langle x, y \rangle \subseteq N_A$, $cn(w) \in \mathbb{C}$ and $cn(w) = 0 \iff w$ is a zero divisor in $A_{\mathbb{C}}$.

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2. If $\langle x, y \rangle \subseteq N_A$, $cn(w) \in \mathbb{C}$ and $cn(w) = 0 \iff w$ is a zero divisor in $A_{\mathbb{C}}$.

Proof of Theorem. Assume $x = \alpha \in \mathbb{R} \Rightarrow S_x = \{x\}$ and $F(\alpha) = F_1(\alpha)$, so

$$\text{if } 0 \neq F(\alpha) = f(x) \Rightarrow CN(F)(\alpha) \neq 0 \text{ and } S_x \cap V(f) = \emptyset \ (1);$$
$$\text{if } 0 = F(\alpha) = f(x) \Rightarrow CN(F)(\alpha) = 0 \text{ and } S_x \subseteq V(f) \ (2).$$

Lemma

Let $w = x + iy \in A_{\mathbb{C}}$, with $y \in N_A \setminus \{0\}$. Let $cn(w) := ww^c$. Then

1. $cn(w) = 0 \iff w = 0$ or $\exists! K \in S_A \mid x + Ky = 0$;
2. If $\langle x, y\rangle \subseteq N_A$, $cn(w) \in \mathbb{C}$ and $cn(w) = 0 \iff w$ is a zero divisor in $A_{\mathbb{C}}$.

Proof of Theorem. Assume $x = \ \ \in \mathbb{R} \implies S_x = \{x\}$ and $F(\ ) = F_1(\ )$, so

$$\text{if } 0 \neq F(\ ) = f(x) \implies CN(F)(\ ) \neq 0 \text{ and } S_x \cap V(f) = \emptyset \ (1);$$

$$\text{if } 0 = F(\ ) = f(x) \implies CN(F)(\ ) = 0 \text{ and } S_x \subseteq V(f) \ (2).$$

Assume now $x \in \ _D \setminus \mathbb{R}$, then

- if $F(z) = F_1(z) + iF_2(z) = 0 \iff F_1(z) = F_2(z) = 0 \implies$ $f(\ + I\ ) = F_1(z) + IF_2(z) = 0\ \forall I \in S_A \implies S_x \subseteq V(f)$ (2).

- if $F(z) \neq 0$, but $CN(F)(z) = 0$, by the previous Lemma $\exists! K \in S_A \mid f(\; + K \;) = F_1(z) + KF_2(z) = 0$ and so $S_x \cap V(f) = \{x + K \;\}$ (3);
- if $F(z) \neq 0$ and $CN(F)(z) \neq 0$, then by the previous Lemma: $f(\; + I \;) = F_1(z) + IF_2(z) \neq 0 \; \forall I \in S_A \implies S_x \cap V(f) = \emptyset$ (1)

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The admissibility of f is necessary in the previous Theorem. Consider again $p(x) = x^2 e_{123} + x(e_1 + e_{23}) + 1$. Note that

$$e_1, e_{23} \in V(p) \cap S_{e_2}, \quad \text{but} \quad p(e_2) \neq 0.$$

Corollary

1. A real slice function has no isolated zeroes;
2. For every admissible slice function it holds

$$V(N(f)) = \bigcup_{x \in V(f)} S_x.$$

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Proof. 1) If $f = I(F)$ is real, $CN(F) = F^2$, so (3) cannot happen.
2)) Let $x = + J \; 2 \; V(f)$, then by the previous Theorem, $CN(F)(z) = 0$, with $z = + i$, but $N(f) = I(CN(F))$, so by (2) of the Theorem, applied to $N(f)$, $S_x \quad V(N(f))$:

$\Leftarrow$ Assume $y = \quad + J \quad \in V(N(f))$ and $z = \quad + i \quad$ so, by the Theorem

$$0 = CN(CN(F))(z) = CN(F)^2(z);$$

since $N(f)$ is real. But then $CN(F)(z) = 0 \implies S_y \cap V(f) \neq \emptyset$, so $y \in S_x$, for some $x \in V(f)$.

Proposition

Let Ω_D be connected and $f \in \mathcal{SR}(\Omega_D)$ admissible.

- If $N(f) \not\equiv 0 \implies C_J \cap \bigcup_{x \in V(f)} \mathbb{S}_x$ is closed and discrete in D_J for each J;
- if $\Omega_D \cap \mathbb{R} \neq \emptyset$, $N(f) \equiv 0 \iff f \equiv 0$.

The Reminder Theorem

De nition

For any y 2 Q_A, the characteristic polynomial of y is the slice regular function on Q_A

$$_y(x) := N(x \quad y) = (x \quad y)(x \quad y^c) = x^2 \quad xt(y) + n(y):$$

Remark. Since y 2 Q_A, $_y$ is real.

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Proposition

Let $y; y^0 \, 2 \; Q_A$, then $\quad _y = \quad _{y^0}$ if and only if $S_y = S_{y^0}$. Moreover, since $\quad _y$ is real, $V(\quad _y) = S_y$.

Theorem (Remainder Theorem)

Let $f \in \mathcal{SR}(\Omega_D)$ be admissible. Let $y \in V(f)$, then:

1. If y is a real zero, then there exists $g \in \mathcal{SR}(\Omega_D)$ so that $f(x) = (x - y)g(x)$.
2. If $y \in \Omega_D \setminus \mathbb{R}$, then there are $h \in \mathcal{SR}(\Omega_D)$ and $a, b \in A$ so that:

$$< a, b > \subset N_A \quad \text{and} \quad f(x) = \Delta_y(x)h(x) + xa + b.$$

In addition:

- y is a spherical zero iff $a = b = 0$;
- y is an $\mathbb{S}_A$ isolated non real zero of f iff $a \neq 0$.

If there is a vector subspace $V \subset N_A$ so that $F(z) \in V \otimes \mathbb{C}$ for every $z \in D$, then g and h are admissible as well. If f is real, g and h are real as well and $a = b = 0$.

Proof. Let $F^- := \frac{1}{2}(F(z) - \overline{F(\bar{z})})$. Then $F = F^+ + F^-$ and this is the unique decomposition satisfying:

- F^- is holomorphic;
- $F^+(\bar{z}) = \overline{F^+(z)}$ and $F^-(\bar{z}) = -\overline{F^-(z)}$.

Assume y is a real zero of $f = I(F)$. Then $F(y) = 0$, so there is an holomorphic map G so that $F(z) = (z - y)G(z)$. In addition, G is complex intrinsic for every $z \neq y$:

$$(\bar{z} - y)G(\bar{z}) = F(\bar{z}) = \overline{F(z)} = (\bar{z} - y)\overline{G(z)}$$

and then over the whole D by continuity. Then G induces a slice regular function satisfying 1.

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Let $y = \alpha + J\beta \in V(f) \setminus \mathbb{R}$ and $\zeta := \alpha + i\beta \in D$. If $F(\zeta) \neq 0$, let $K \in \mathbb{S}_A$ be the unique value s.t $F_1(\zeta) + KF_2(\zeta) = 0$, otherwise let K be any element of $\mathbb{S}_A$. Define $F_K := (1 - iK)F$.

Observe that F_K is holomorphic and vanishes at ζ:

$$F_K(\zeta) = F_1(\zeta) + iF_2(\zeta) \quad iKF_1(\zeta) + KF_2(\zeta) = (1 \quad iK)(F_1(\zeta) + KF_2(\zeta)) = 0.$$

Hence, there is an holomorphic G such that $F_K(z) = (z \quad \zeta)G(z)$. Let G_1 be the holomorphic map s.t. $G(z) \quad G(\bar{\zeta}) = (z \quad \bar{\zeta})G_1(z)$, then:

$$\begin{aligned} F_K(z) &= (z \quad \zeta)G(z) = (z \quad \zeta)((z \quad \bar{\zeta})G_1(z) + G(\bar{\zeta})) \\ &= \quad {}_y(z)G_1(z) + (z \quad \zeta)G(\bar{\zeta}). \end{aligned}$$

Where $\quad {}_y(z) = z^2 \quad zt(y) + n(y)$ is the complex intrinsic polynomial inducing $\quad {}_y(x)$. Observe that:

$$F_K^+(z) = \frac{1}{2}(F_K(z+) + \overline{F(\bar{z})}) = \frac{1 \quad iK}{2}F(z) + \frac{1 + iK}{2}\overline{F(\bar{z})} = F(z)$$

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Then it follows:

$$F(z) = F_K^+(z) = \quad {}_y(z)G_1^+(z) + \frac{1}{2}\ \ (z \quad \zeta)G(\bar{\zeta}) + (z \quad \bar{\zeta})\overline{G(\bar{\zeta})}$$

$$= \quad {}_y(z)H(z) + za + b,$$

where $H(z) = G_1^+(z)$, which is holomorphic and complex intrinsic, and $a, b\ 2\ A$. Since $F_1(\bar{\zeta}) + iF_2(\bar{\zeta}) = F(\bar{\zeta}) = \zeta a + b$ we get:

$$a = \beta \ \ ^1F_2(\bar{\zeta}) \quad \text{and} \quad b = F_1(\bar{\zeta}) \quad \alpha\beta \ \ ^1F_2(\bar{\zeta}),$$

then $a, b\ 2\ N_A$, since f is admissible. This proves 2.

Then it follows:

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where $H(z) = G_1^+(z)$, which is holomorphic and complex intrinsic, and $a, b \in A$. Since $F_1(\bar{\zeta}) + iF_2(\bar{\zeta}) = F(\bar{\zeta}) = \zeta a + b$ we get:

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then $a, b \in N_A$, since f is admissible. This proves 2.

Remark. When A = H, O it was proved that the statement 2 can be rephrased in terms of 1.

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Corollary

Let $f\ 2\ SR(\ _D)$ *be admissible. If* $\mathbb{S}_y$ *contains at least one zero of* f*, of whatever type, then* $\ _y$ *divides* $N(f)$*.*

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Corollary

Let f 2 $SR(\ _D)$ be admissible. If S_y contains at least one zero of f, of whatever type, then $\ _y$ divides $N(f)$.

Recall that:

$$f \text{ admissible } \quad) \quad N(f) \text{ real } \quad) \quad N(f) = \ ^s_y g, \text{ for } g \text{ real.}$$

An equivalent version comes from the division of the holomorphic functions F and G inducing f and g.

Definition

Let $f \in SR(\Omega_D)$ be admissible, with $N(f) \not\equiv 0$. Then $y \in V(f)$ is a zero of multiplicity s if $\Delta_y^s \mid N(f)$ but $\Delta_y^{s+1} \nmid N(f)$. The integer s is called multiplicity of y and is denoted by $m_f(y)$.

Remarks.

- if y is real, $m_f(y) = s$ if and only if $(x - y)^s \mid f$ and $(x - y)^{s+1} \nmid f$;
- if y is a spherical zero, then Δ_y divides both f and f^c, indeed:

$$f = \Delta_y h \quad \Rightarrow \quad f^c = \Delta_y h^c.$$

  Hence $m_f(y) \geq 2$;
- if $m_f(y) = 1$, y is called a simple zero of f.

The Fundamental Theorem of Algebra

Let $p(x) = \sum_{j=0}^{m} x^j a_j$ be an admissible slice regular polynomial of degree m with coefficients in $a_j \in A$. The normal polynomial of p is the slice regular polynomial $N(p)$ and it coincides with:

$$(p \ast p^c)(x) = \sum_{n=1}^{2m} x^n \Big(\sum_{j+k=n} a_j a_k^c \Big) =: \sum_{n=1}^{2m} x^n c_n.$$

Observe that $N(p)$ is a real polynomial of degree $2m$.

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The Fundamental Theorem of Algebra

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$$(p * p^c)(x) = \sum_{n=1}^{2m} x^n \Big(\sum_{j+k=n} a_j a_k^c \Big) =: \sum_{n=1}^{2m} x^n c_n.$$

Observe that $N(p)$ is a real polynomial of degree $2m$.

Remark. A polynomial is real (as a slice function) if and only if its coefficients are real. Indeed, consider $P(z) = \sum_{j=0}^{m} z^j a_j$, then

$$P(0) = a_0 \in \mathbb{C} \cap A = \mathbb{R}.$$

and so on with the other coefficients.

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Theorem (Fundamental Theorem of Algebra)

Let $p(x) = \sum_{j=0}^{m} x^j a_j$ be an admissible polynomial of degree $m > 0$ with coefficients in A. Then $V(p) := \{x \in Q_A \mid p(x) = 0\}$ is non-empty. More precisely, there are distinct $\mathbb{S}_{x_1}, \dots, \mathbb{S}_{x_t}$ s.t

$$V(p) \subseteq \bigcup_{k=1}^{t} \mathbb{S}_{x_k} = V(N(p)), \quad V(p) \cap \mathbb{S}_{x_k} \neq \emptyset, \quad \text{for } k = 1, \dots, t,$$

and for any choice of zeros $y_1 \in \mathbb{S}_{x_1}, \dots, y_t \in \mathbb{S}_{x_t}$ of p, it holds:

$$\sum_{k=1}^{t} m_p(y_k) = m.$$

Remark. If $< a_0, \dots, a_m > \subseteq N_A$, then $p(x) = \sum_{j=1}^{m} x^j a_j$ is admissible.

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Proof. STEP 1: $V(p) \neq \emptyset$.

Since p is admissible $N(p)$ is real, so $N(p)$ is a polynomial with real coefficients and degree $2m$. Then, for every $J \in \mathbb{S}_A$

$$V(N(p))_J = \mathbb{C}_J \cap \bigcup_{y \in V(p)} \mathbb{S}_y \neq \emptyset$$

Since $N(p)$ is real, $V_J(N(p)) \cap \mathbb{S}_y \neq \emptyset$ if and only if $V(p) \cap \mathbb{S}_y \neq \emptyset$, then $V(p) \neq \emptyset$.

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STEP 2: $m_{N(p)}(y) = 2m_p(y)$ for every $y \in V(p)$.

Let $g \in \mathcal{SR}(Q_A)$ so that $N(p) = \Delta_y^s g$ and $\Delta_y \nmid g$, then g is real since $N(p)$ and Δ_y are so and $s = m_p(y)$. Then:

$$N(N(p)) = N(p)N(p)^c = N(p)^2 = \Delta_y^2 g^2 \quad \Rightarrow \quad m_{N(p)}(y) \geq 2m_p(y).$$

We need to prove that $\Delta_y \nmid g^2$.

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Assume $g^2 = \ _y h$, with $h\ 2\ SR(Q_A)$ real. Let $g = I(G)$ and $h = I(H)$, then $G(z)^2 = \ _y(z)H(z)$ on D. Let $y = \alpha + \beta J$ and $\zeta = \alpha + i\beta$. Then:

$$G(\zeta)^2 = \ _y(\zeta)H(\zeta) = 0 \quad) \quad G(\zeta) = 0,$$

that is $g(y) = 0$. But, since g is real, the Reminder Theorem implies that $_y\ j\ g$, which is a contradiction.

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that is $g(y) = 0$. But, since g is real, the Reminder Theorem implies that $_y \; j \; g$, which is a contradiction.

STEP 3: $\mathrm{P}_{k=1}^{t} m_p(y_k) = m.$

If y is a real zero of $N(p)$, then $m_{N(p)_J}(y) = m_{N(p)}(y)$. Let $y = \alpha + \beta J$ be a spherical zero, hence it corresponds to a non-real zero of $N(p)_J$ and $N(p)(y^c) = 0$ as well. In addition, since $_y(x) = (x \quad y) \quad (x \quad y^c)$, we get that $m_{N(p)}(y) = m_{N(p)_J}(y) + m_{N(p)_J}(y^c)$. Therefore:

$$2m = \mathrm{X}_{k=1}^{t} m_{N(p)}(y_k) = 2 \mathrm{X}_{k=1}^{t} m_p(y_k).$$

Examples

Consider the Clifford algebra $\mathbb{R}_3$. Then:

1. Admissibility is necessary for studying zeros in Q_A. The polynomial $p(x) = xe_{23} + e_1$ only vanishes at $y = e_{123} \notin Q_A$. But p is not admissible, indeed $e_1 + e_{23} \notin N_A$.

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2. The Fundamental Theorem of Algebra works completely only over Q_A, even for non-spherical zeros. The polynomial $p(x) = x^2 - 1$ has four roots $\{\pm 1, \pm e_{123}\}$, two of them in Q_A.

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Slice regular functions on real alternative algebras