Real Algebraic Geometry and Singularities

Conference in honor of Wojciech Kucharz's 70th birthday

The topology of real algebraic sets with isolated singularities is determined by the field of rational numbers

Enrico Savi (University of Trento)

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Schedule of the talk:

1. Motivation
2. Notions of real $\mathbb{Q}$-algebraic geometry
3. 1. Relative Weierstrass approximation theorem
   2. The compact smooth case
   3. The relative compact smooth case
4. Isolated singularities

1. Motivation

A deep application of Zariski-equisingularity leads to:

THEOREM: (A. Parusiński - G. Rond, 2020) Let $X \subset \mathbb{R}^n$ be a real algebraic set. Then there exists a real algebraic set $X' \subset \mathbb{R}^n$ and an homeomorphism $h: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with the following properties:

1. h maps X onto X';
2. X' is defined over $\mathbb{R}_{alg} := \overline{\mathbb{Q}}^r$;
3. h can be chosen subanalytic and arc-analytic.

What happens when replacing $\overline{\mathbb{Q}}$ with $\mathbb{Q}$?

The Theorem only applies when the extension of $\mathbb{Q}$ obtained by adding the coefficients of generators of $\mathcal{I}(X)$ is purely Transcendental.

Open question:

Is every real algebraic set $X \subset \mathbb{R}^n$ homeomorphic to some real algebraic subset $X' \subset \mathbb{R}^m$ which is defined over $\mathbb{Q}$?

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Is every real algebraic set $X \subset \mathbb{R}^n$ homeomorphic to some real algebraic subset $X' \subset \mathbb{R}^m$ which is defined over $\mathbb{Q}$?

2. Notions of real $\mathbb{Q}$-algebraic geometry

Let $X \subset \mathbb{R}^n$, define the ideal:

$$\mathcal{I}_{\mathbb{Q}}(X) := \{ q \in \mathbb{Q}[x_1, \dots, x_n] : q(x) = 0 \quad \forall x \in X \}.$$

DEFINITION: Let $X \subset \mathbb{R}^n$ be an algebraic set. We say that:

1. X is $\mathbb{Q}$-algebraic if
$$X = \mathcal{Z}(\mathcal{I}_{\mathbb{Q}}(X))$$
3. X is defined over $\mathbb{Q}$ in $\mathcal{R}(\mathbb{R}^n)$ if
$$\mathcal{I}^r_{\mathbb{R}^n}(X) = \mathcal{I}_{\mathbb{Q}}(X)\mathcal{R}(\mathbb{R}^n)$$
4. X is defined over $\mathbb{Q}$ if
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Let $a = (a_1, _, a_n) \in \mathbb{R}^n$, denote by

$$\mathfrak{n}_{\mathbb{R},a} := (x_1 - a_1, __, x_n - a_n) .$$

DEFINITION: (J. Fernando, R. Ghiloni)

Define the $\mathbb{R}|\mathbb{Q}$-local ring and $\mathbb{R}|\mathbb{Q}$-regular points as

$$\mathcal{R}^{\mathbb{R}|\mathbb{Q}}_{X,a} := \mathbb{R}[x_1, _, x_n]_{\mathfrak{n}_{\mathbb{R},a}} / \mathcal{I}_{\mathbb{Q}}(X)\, \mathbb{R}[x_1, __, x_n]_{\mathfrak{n}_{\mathbb{R},a}}$$

$$\mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(X) := \{ a \in X : \mathcal{R}^{\mathbb{R}|\mathbb{Q}}_{X,a} \text{ is regular of dimension } \dim X \}$$

Let X be a $\mathbb{Q}$-algebraic set, we say That

2. X is $\mathbb{Q}$-determined if $\mathrm{Reg}(X) = \mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(X)$.

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Equivalences:

LEMMA: let $X \subset \mathbb{R}^n$ be a nonsingular $\mathbb{Q}$-algebraic set of dimension d, then the following are equivalent:

(i) X is $\mathbb{Q}$-determined;

(ii) for every $a \in \mathrm{Reg}(X)$ there are $q_1, \dots, q_{n-d} \in \mathcal{I}_{\mathbb{Q}}(X)$ s.t.

(a) $\nabla q_1(a), \dots, \nabla q_{n-d}(a)$ are lin. independent,

(b) There is a $\mathcal{Z}$-open set U of $\mathbb{R}^n$ s.t.

$$X \cap U = \mathcal{Z}(q_1, \dots, q_{n-d}) \cap U .$$

(iii) $\mathcal{I}^r_{\mathbb{R}^n \setminus \mathrm{Sing}(X)}(\mathrm{Reg}(X)) = \mathcal{I}_{\mathbb{Q}}(X)\, \mathcal{R}(\mathbb{R}^n \setminus \mathrm{Sing}(X))$

i.e. X is defined over $\mathbb{Q}$ in $\mathcal{R}(\mathbb{R}^n)$.

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3.1 Relative Weierstrass approximation theorem

THEOREM: Let $L \subset \mathbb{R}^n$ be a compact nonsingular $\mathbb{R}$-determined algebraic set, let $L \subset K$ be a compact neighborhood of L and $f: (\mathbb{R}^n, L) \longrightarrow (\mathbb{R}, 0)$ smooth.

Then f can be arbitrarily $\mathcal{C}^\infty$-approximated on K by some polynomial $p \in \mathcal{I}_{\mathbb{R}}(L)$.

Sketch of the proof: Let $d := \dim(L)$ and $q_0, q_1, \dots, q_\ell \in \mathcal{I}_{\mathbb{R}}(L)$ s.t.

$$L = \mathcal{Z}(q_0) \quad \text{and} \quad \operatorname{rnk}\left(J(q_1, \dots, q_\ell)(a)\right) = n - d \quad \forall a \in L.$$

1. For every $a \in \mathbb{R}^n$ there are a neighborhood U_a of a in $\mathbb{R}^n$ and $u_0, \dots, u_\ell \in \mathcal{C}^\infty(U_a)$ s.t. $f|_{U_a} = \sum_{i=0}^{\ell} u_i q_i$.

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- if $a \in L \longrightarrow$ use the $n-d$ independent gradients at a,

- if $a \in \mathbb{R}^n \setminus L \longrightarrow$ fix $U := \mathbb{R}^n \setminus Z(q_0)$, $u_0 := \frac{f}{q_0}$

2. Paste together the local u_i getting global $v_i \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that:

$$f = \sum_{i=0}^{l} v_i \, q_i$$

3. $\mathcal{C}^\infty$-approximate each v_i with a polynomial $w_i \in \mathbb{Q}[x_1, \dots, x_n]$ on K. Then, $q := \sum_{i=0}^{l} w_i \, q_i \in I_{\mathbb{Q}}(L)$ is a $\mathcal{C}^\infty$-approximation of f on K.

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Then, further approximation techniques by Tognoli - Akbulut - King can be developed over $\mathbb{Q}$.

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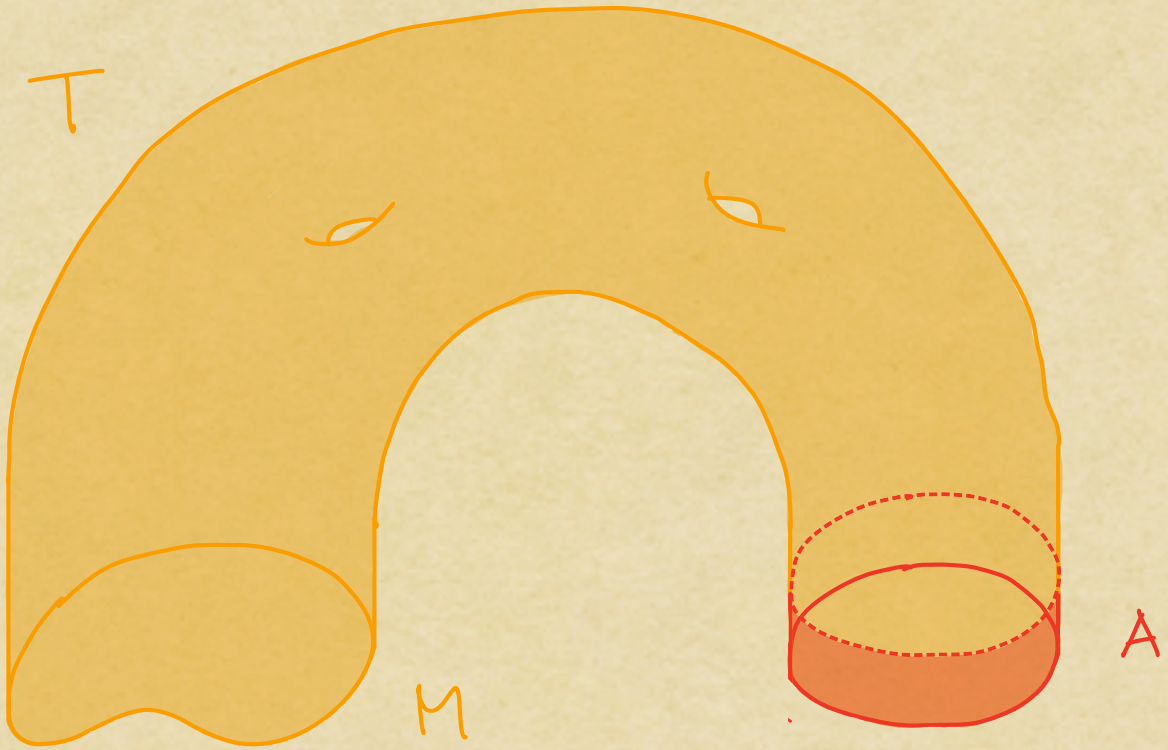
Then, further approximation techniques by Tognoli - Akbulut - King can be developed over $\mathbb{Q}$.

Our result:

THEOREM: (R. Ghiloni, E.S.) Every compact smooth manifold $M \subset \mathbb{R}^m$ is diffeomorphic to some nonsingular $\mathbb{R}$-determined algebraic subset $X \subset \mathbb{R}^{m+n}$ and the diffeomorphism can be chosen arbitrarily $\mathcal{C}^\infty_W$-close to the inclusion map $\imath : M \hookrightarrow \mathbb{R}^{m+n}$.

If in addition $M \subset \mathbb{R}^m$ is algebraic, then the diffeomorphism can be chosen to be a Nash diffeomorphism.

M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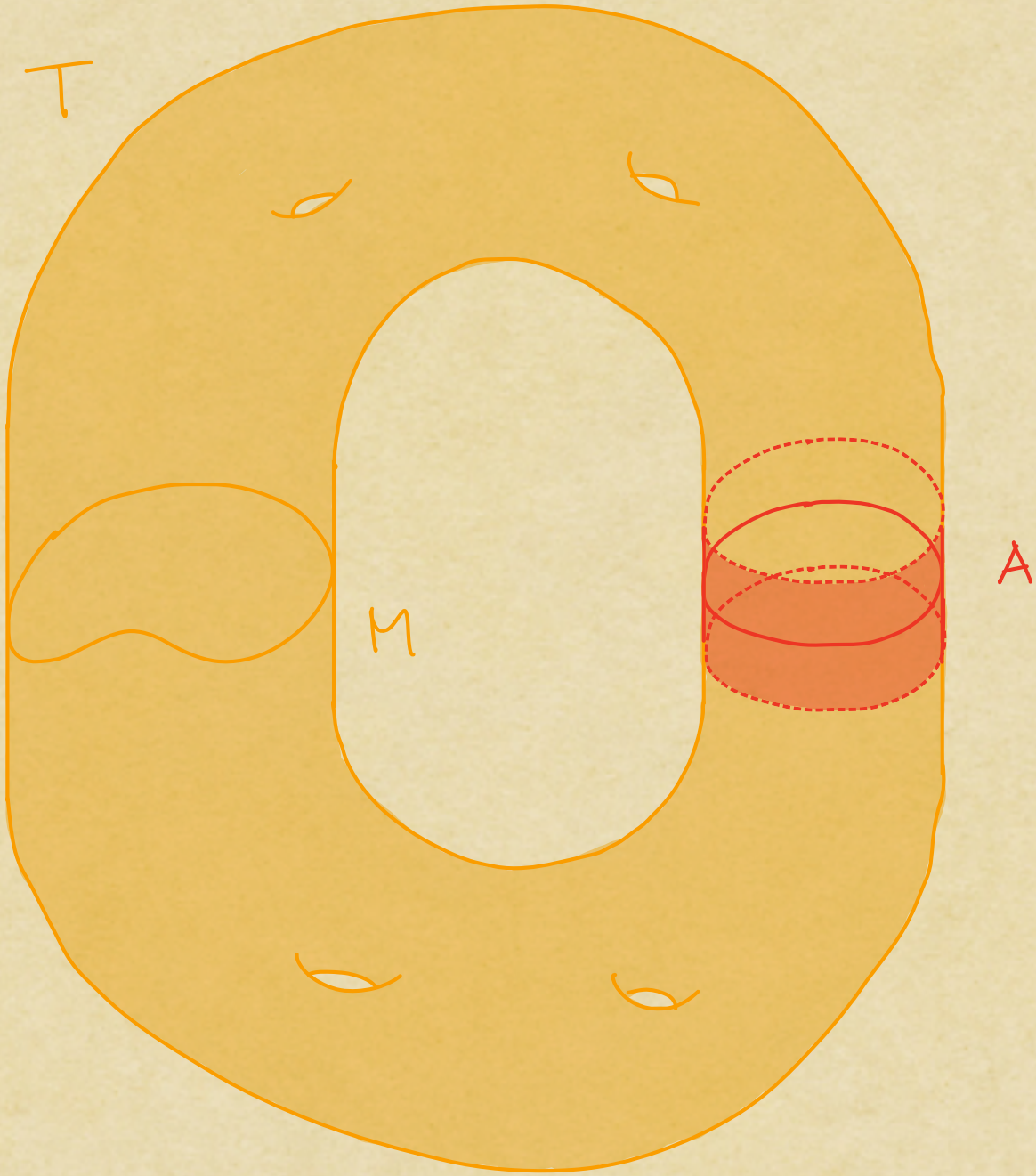
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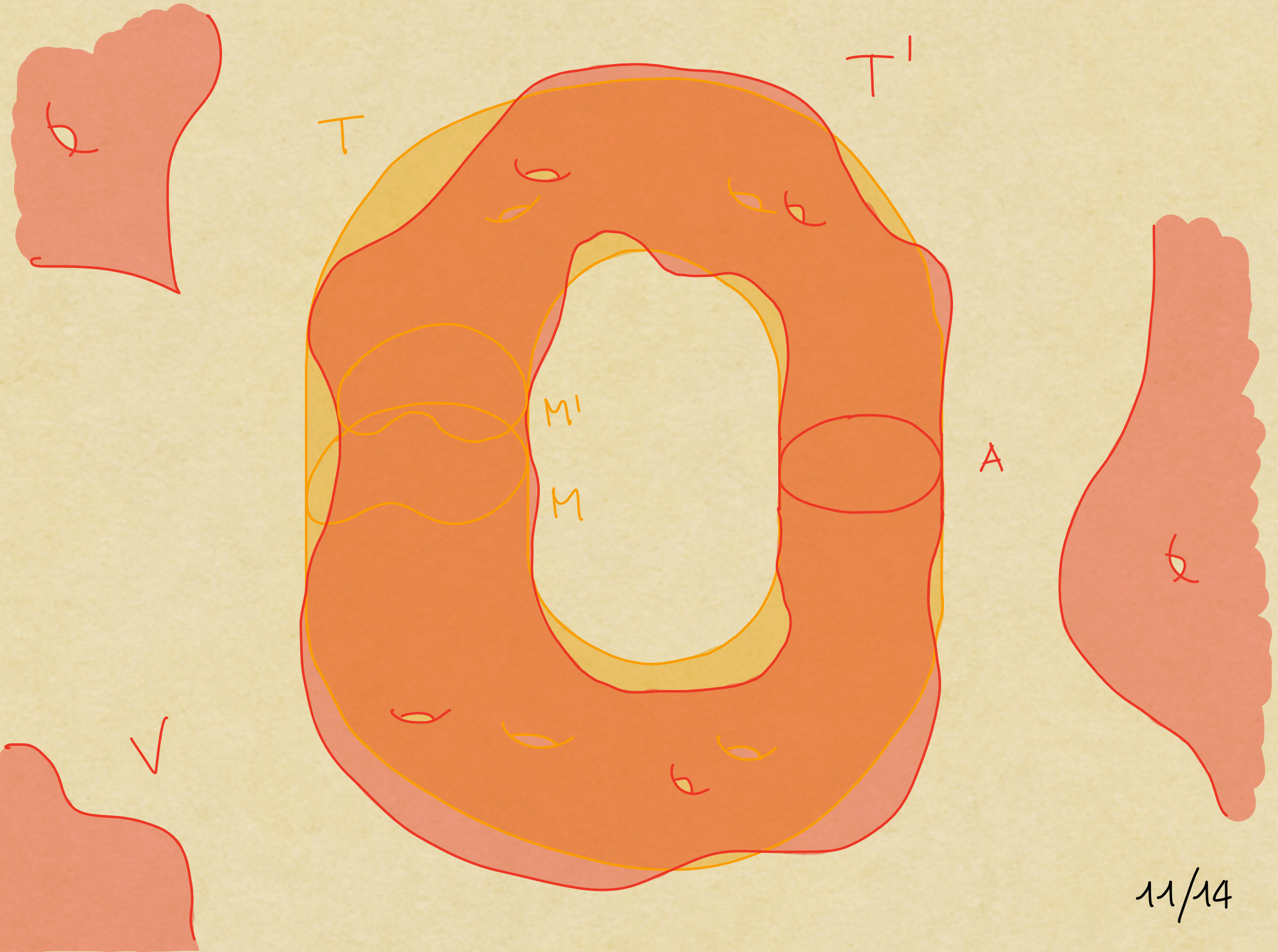
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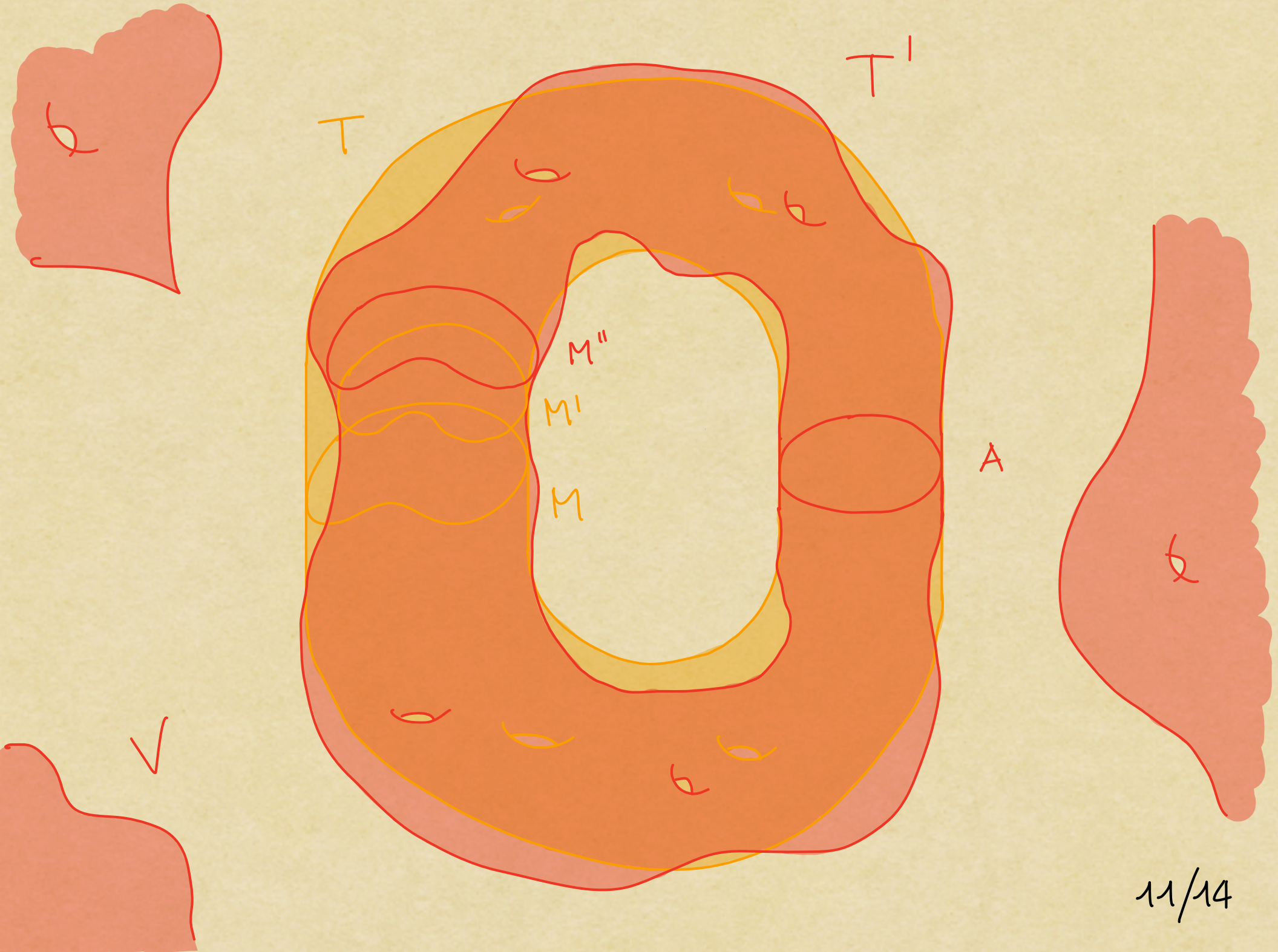
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3.2 The relative compact smooth case

THEOREM: (R. Ghiloni, E.S.) Every compact smooth manifold $M \subset \mathbb{R}^m$ with hypersurfaces $\{M_i\}_{i=1}^{\ell}$ in general position is diffeomorphic to some nonsingular $\mathbb{Q}$-determined algebraic subset $X \subset \mathbb{R}^{m+n}$ with nonsingular $\mathbb{Q}$-determined algebraic hypersurfaces $\{X_i\}_{i=1}^{\ell}$ in general position s.t. the diffeomorphism carries each M_i to X_i and it can be chosen $\mathcal{C}^\infty_W$-close to $z : M \longrightarrow \mathbb{R}^{m+n}$.

If in addition M and the M_i's are algebraic, then the diffeomorphism can be chosen to be a Nash diffeomorphism.

4. Isolated singularities

The following is our main result :

THEOREM : (R. Ghiloni, E.S.) Let $V \subset \mathbb{R}^n$ be a compact real algebraic set with isolated singularities. There exists a real algebraic set $V' \subset \mathbb{R}^p$ and a semialgebraic homeomorphism $h : V \longrightarrow V'$ s.t.

1. V' is $\mathbb{Q}$-determined ;
2. h extends to an homeomorphism $\mathbb{R}^p \longrightarrow \mathbb{R}^p$;
3. $h(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$ and $h|_{\mathrm{Reg}(V)}$ is a Nash diffeomorphism ;
4. h can be chosen arbitrarily $\mathcal{C}^0_W$-close to $\imath : V \hookrightarrow \mathbb{R}^p$ and $\mathcal{C}^\infty_W$-close to $\imath$ over any compact subset K of $\mathrm{Reg}(V)$.

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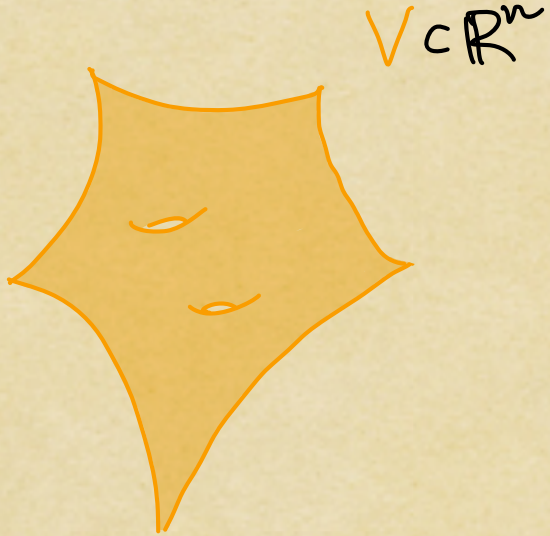
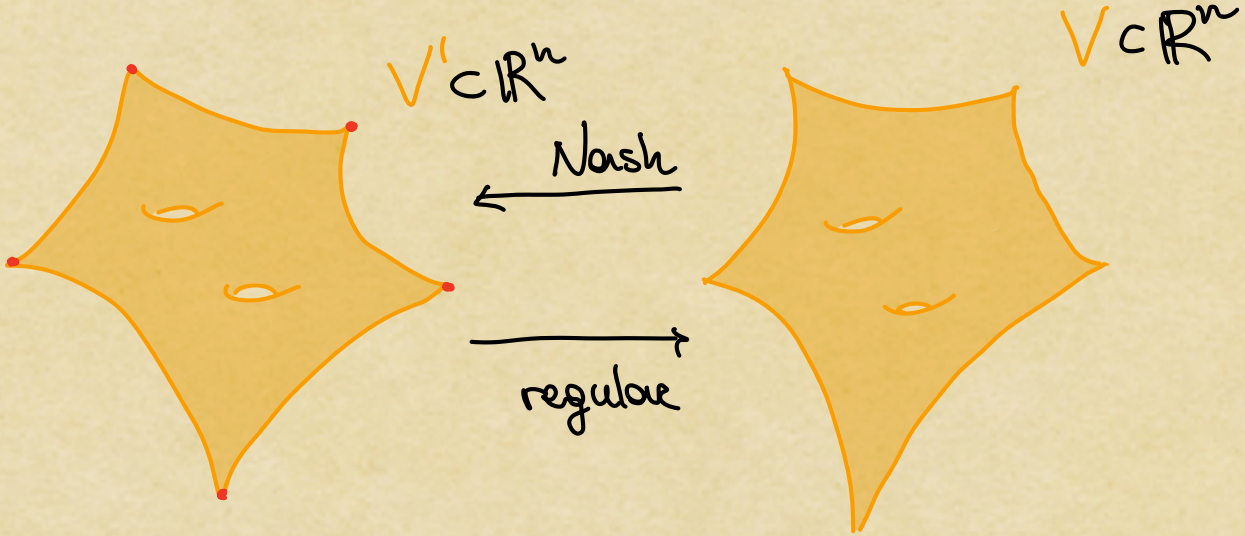
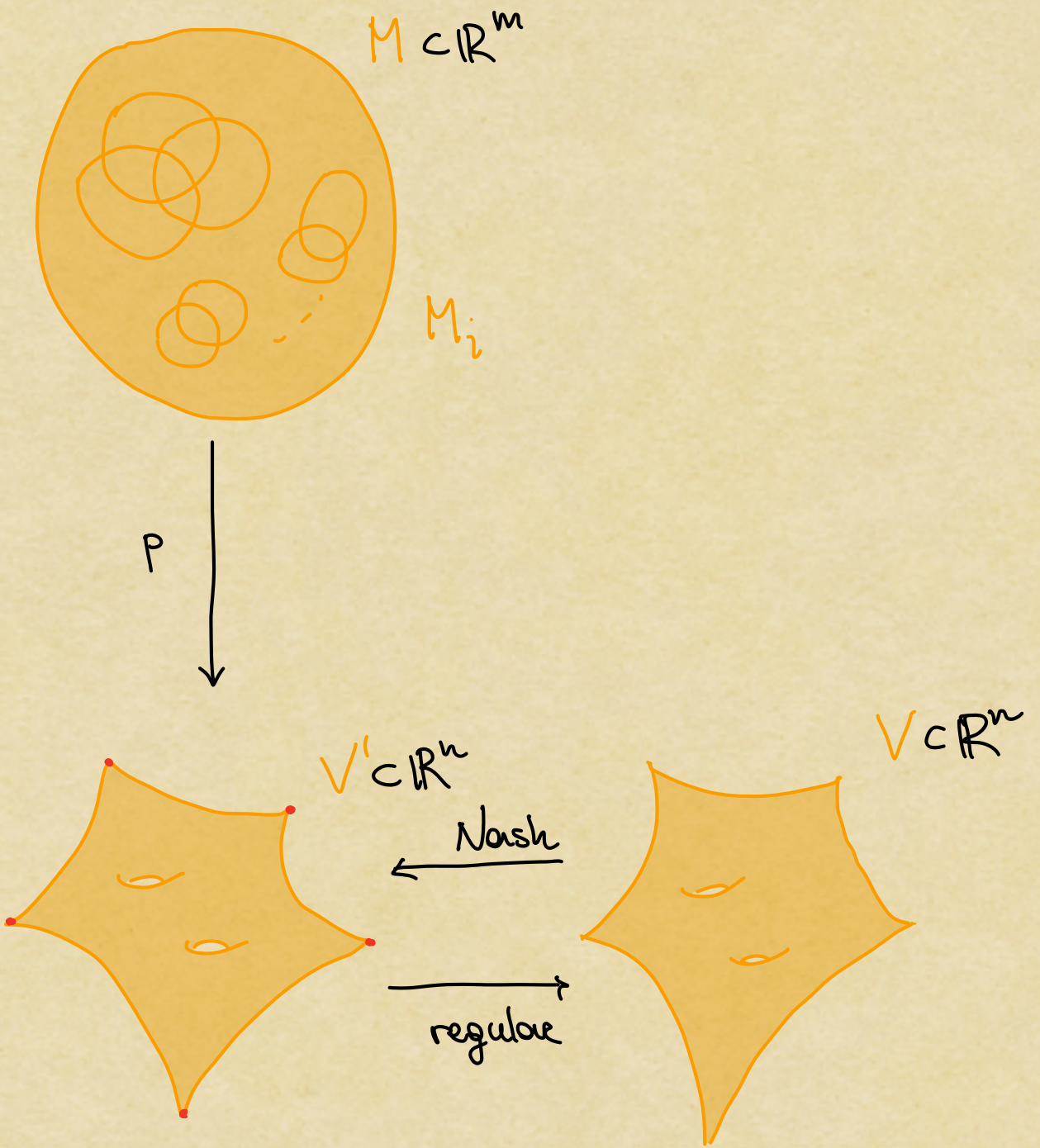

V ⊂ ℝⁿ


$V' \subset \mathbb{R}^n$
$V \subset \mathbb{R}^n$
Nash
regular


$M \subset \mathbb{R}^m$
M_i
p
$V' \subset \mathbb{R}^n$
Nash
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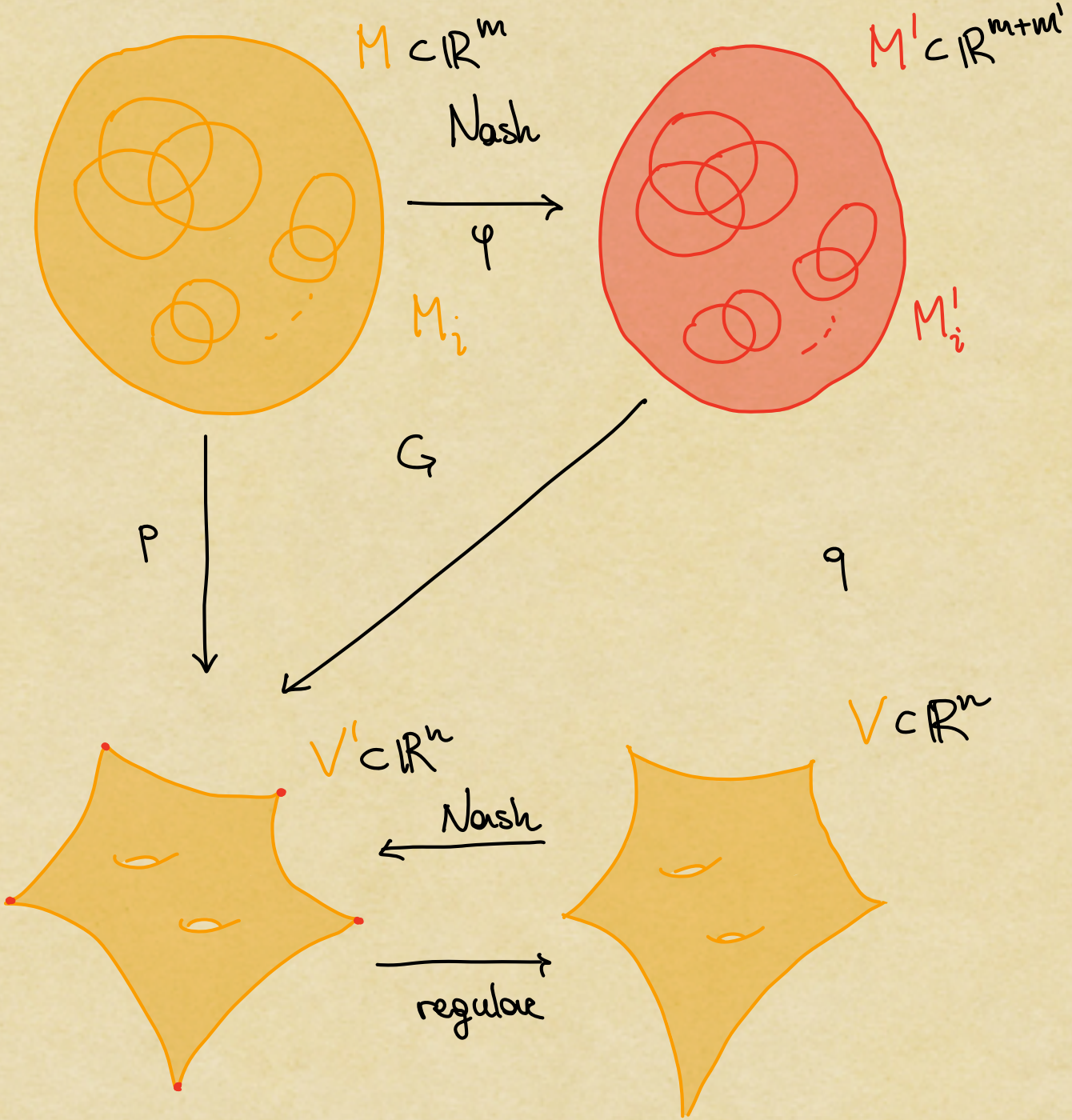
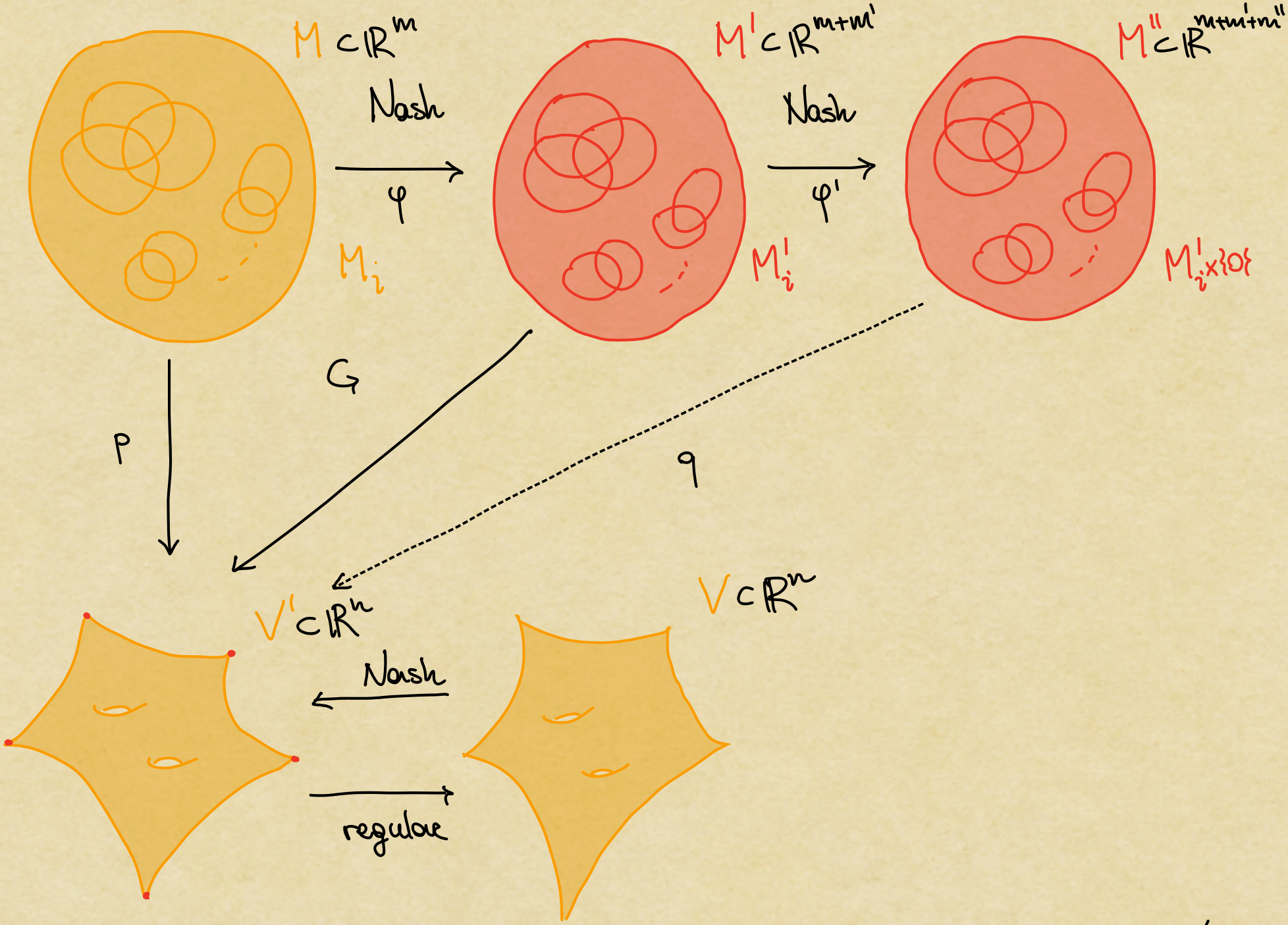
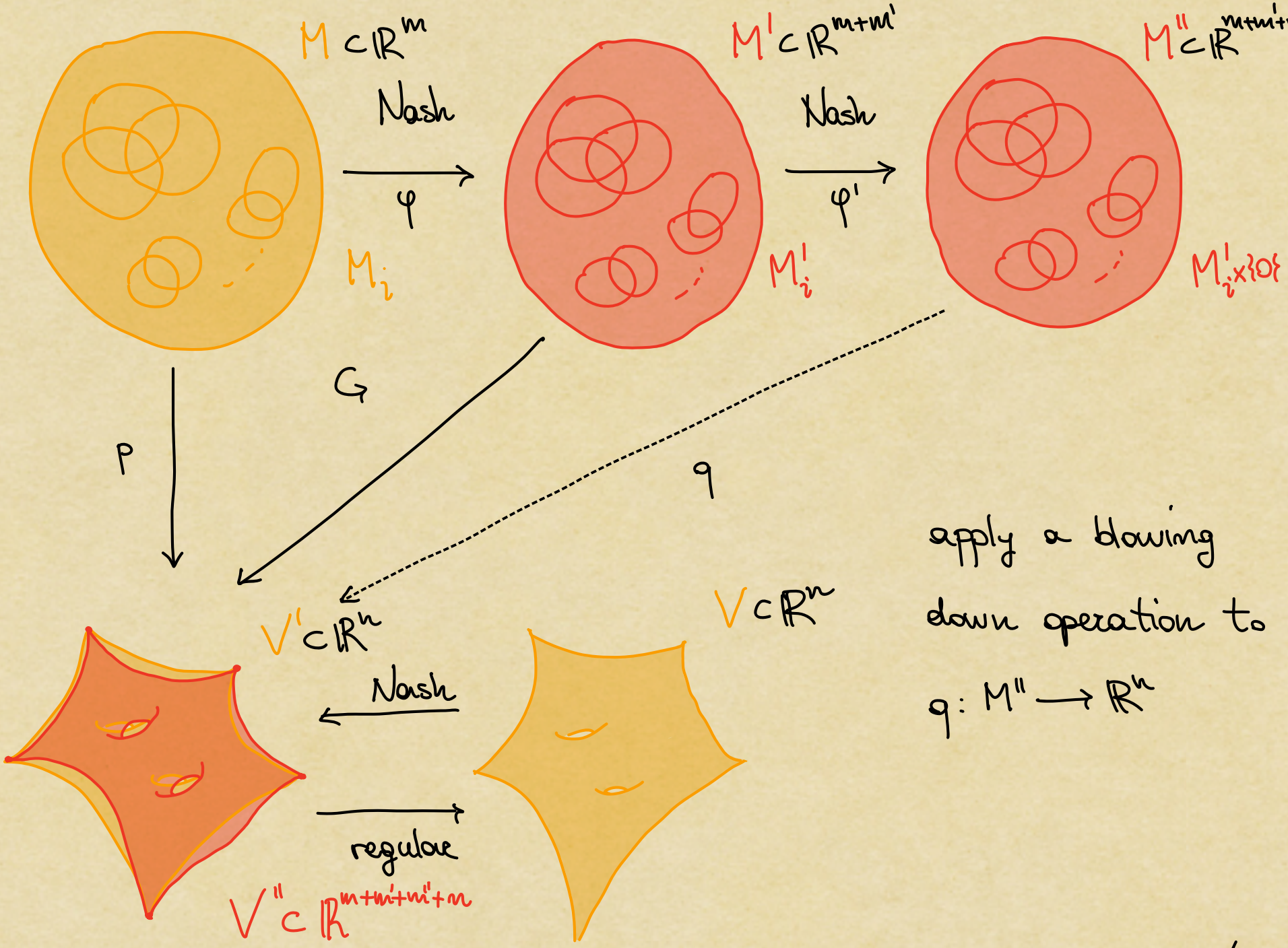
$V \subset \mathbb{R}^n$


M ⊂ ℝ^m
Nash
φ
M_i
M' ⊂ ℝ^{m+m'}
M'_i
G
p
q
V' ⊂ ℝ^n
Nash
V ⊂ ℝ^n
regulue


$M \subset \mathbb{R}^m$
Nash
φ
M_i
$M' \subset \mathbb{R}^{m+m'}$
Nash
φ'
M'_i
$M'' \subset \mathbb{R}^{m+m'+m''}$
$M'_{i \times \{0\}}$
G
p
q
$V' \subset \mathbb{R}^n$
$V \subset \mathbb{R}^n$
Nash
regular



apply a blowing down operation to

$q : M'' \longrightarrow \mathbb{R}^n$

COROLLARY: (R. Ghiloni, E.S.) Let $V \subset \mathbb{R}^n$ be a ~~compact~~ real algebraic set with isolated singularities. There exists a real algebraic set $V' \subset \mathbb{R}^p$ and a semialgebraic homeomorphism $h: V \longrightarrow V'$ s.t.

1. V' is $\mathbb{Q}$-determined ;
2. h extends to an homeomorphism $\mathbb{R}^p \longrightarrow \mathbb{R}^p$;
3. $h(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$ and $h|_{\mathrm{Reg}(V)}$ is a Nash diffeomorphism ;
4. h can be chosen arbitrarily $\mathcal{C}^0_W$-close to $i: V \hookrightarrow \mathbb{R}^p$ over any compact subset L of V and $\mathcal{C}^\infty$-close to i over any compact subset K of $\mathrm{Reg}(V)$.

COROLLARY: (R. Ghiloni, E.S.) Let $V \subset \mathbb{R}^n$ be an algebraic set.

If $(V,0)$ is an isolated singularity, then there are a $\mathbb{Q}$-determined algebraic set $V' \subset \mathbb{R}^m$, containing 0, and a semialgebraic homeomorphism:

$$h : (U,0) \longrightarrow (U',0),$$

with U and U' neighbourhoods of 0 in V and V'.

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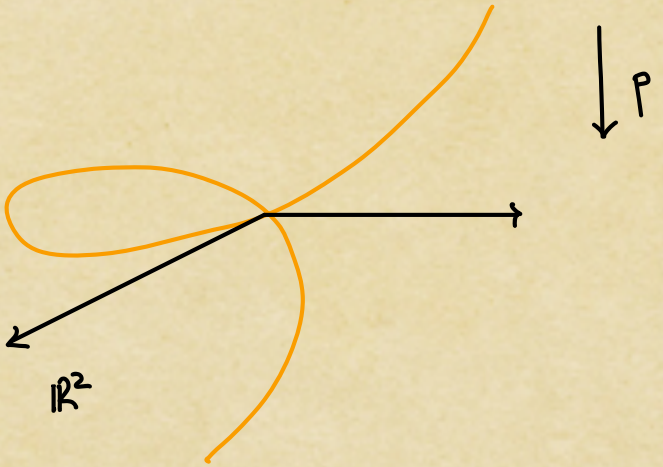

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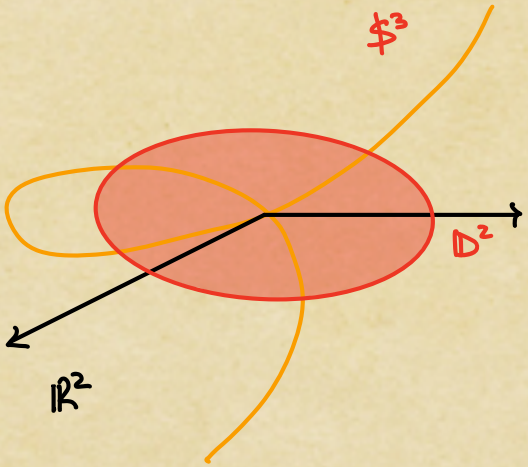

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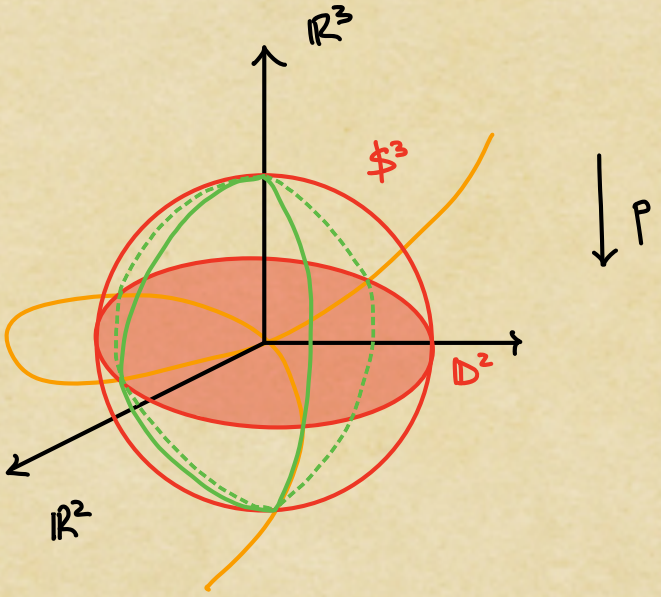

COROLLARY: (R.Ghiloni, E.S.) Let $V \subset \mathbb{R}^u$ be an algebraic set.

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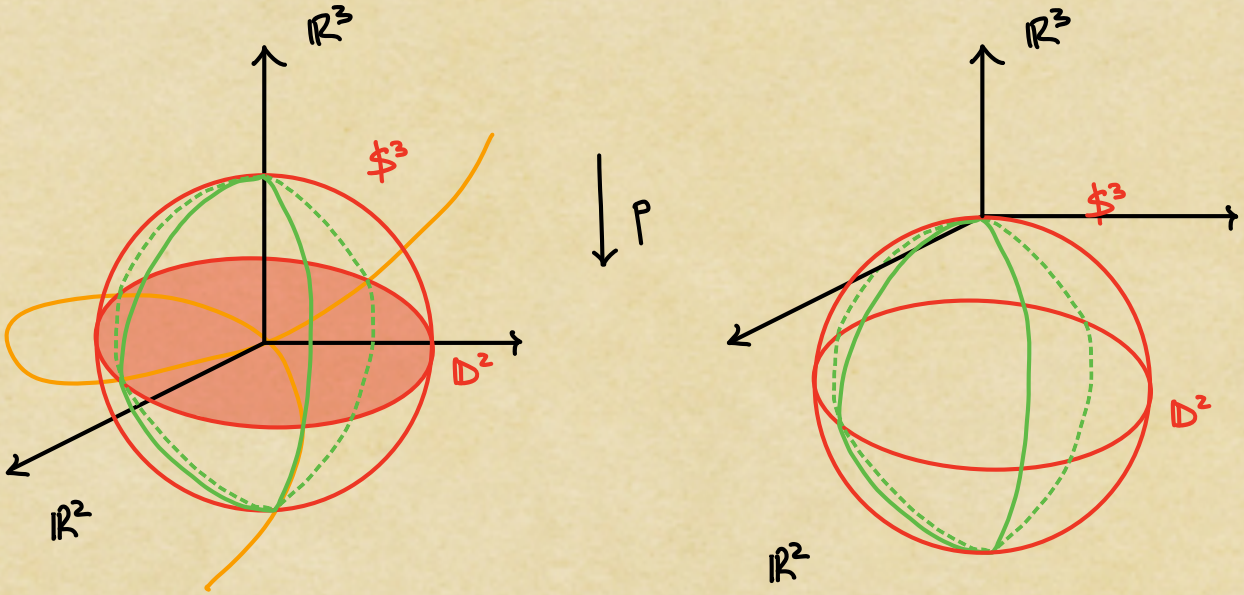

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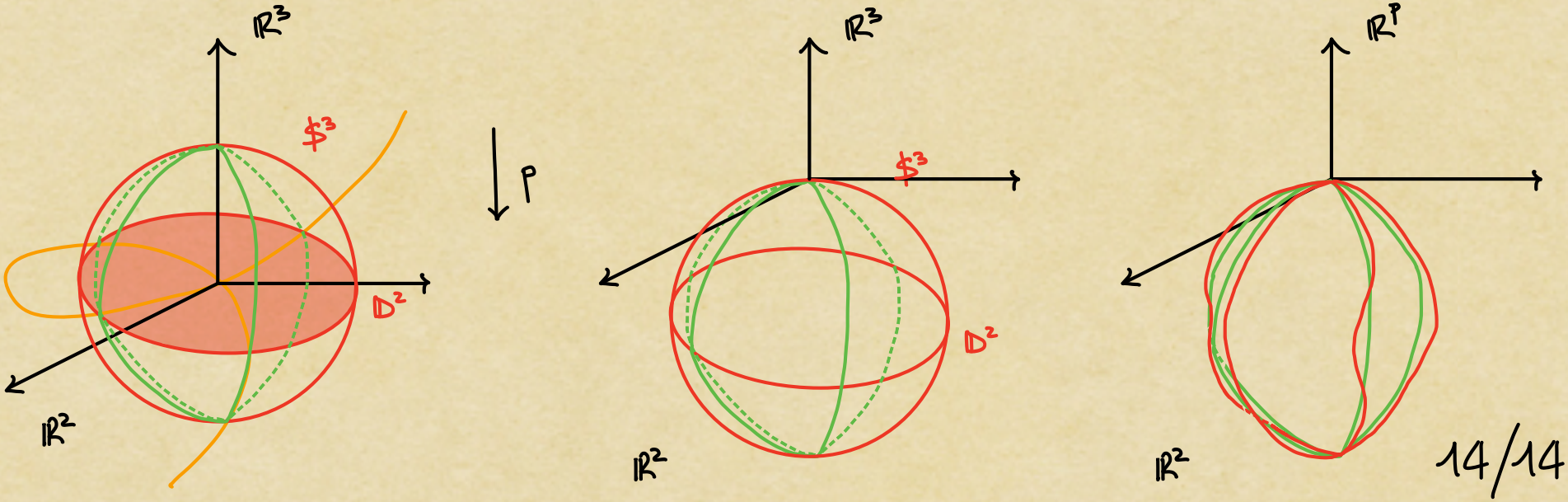
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