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The topology of real algebraic sets with isolated singularities is determined by the field of rational numbers

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Motivation

In what follows $V \subset \mathbb{R}^n$ is a (real) algebraic set. A very natural question is: Can we find another algebraic set V', homeomorphic to V, whose equations are simpler? A first answer is given by A. Parusinski and G. Rond in [6]. Making use of Zariski equisingularity, they are able to prove the following result.

Theorem 1 ([6, Theorem 1]). *There exists an algebraic set $V' \subset \mathbb{R}^n$ and a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that:*

1. *V' is defined over $\overline{\mathbb{Q}}^r := \overline{\mathbb{Q}} \cap \mathbb{R}$,*
2. *$h(V) = V'$,*
3. *h is sub-analytic and arc-analytic.*

If, in addition, the field extension of $\mathbb{Q}$ generated by the coefficients of the polynomial equations defining V is purely transcendental, then V' in Theorem 1 can be chosen to be defined over $\mathbb{Q}$, see [6, Remark 13]. However, in the general setting the procedure to get coefficients over $\mathbb{Q}$ fails because of lack (in general) of rational points in algebraic sets. Thus, in [5, Question 3.4] the following question is stated:

Open Question: Is there any $m \geq n$ and $V' \subset \mathbb{R}^m$ such that V' is defined over $\mathbb{Q}$ and V' is homeomorphic to V?

Real $\mathbb{Q}$-algebraic geometry

Let us recall some notions of real $\mathbb{Q}$-algebraic geometry [3, 4]. Denote by $\mathbb{Q}[x] := \mathbb{Q}[x_1, \dots, x_n]$. Let $V \subset \mathbb{R}^n$ be an algebraic set and let $F \subset \mathbb{Q}[x]$. Define:

$$\mathcal{I}_{\mathbb{Q}}(V) := \{q \in \mathbb{Q}[x] \,|\, q(x) = 0,\ \forall x \in V\},$$
$$\mathcal{Z}(F) := \{x \in \mathbb{R}^n \,|\, q(x) = 0,\ \forall q \in F\}.$$

Definition 2 ([3]). *We say that V is* $\mathbb{Q}$-algebraic *if it is the common solution set of a system of polynomial equations with rational coefficients. Namely, if*

$$V = \mathcal{Z}(\mathcal{I}_{\mathbb{Q}}(V)).$$

Remark. $\mathbb{Q}$-algebraic subsets of $\mathbb{R}^n$ are the closed subsets of a topology strictly coarser that the usual Zariski topology of $\mathbb{R}^n$.

Let $a \in V$. Denote by $\mathfrak{n}_a$ the maximal ideal of $\mathbb{R}[x]$ of all real polynomials vanishing at a.

Definition 3 ([3]). *Suppose V is $\mathbb{Q}$-algebraic. We define the* $\mathbb{R}|\mathbb{Q}$-local ring of V at a *as the local ring*

$$\mathcal{R}^{\mathbb{R}|\mathbb{Q}}_{V,a} := \mathbb{R}[x]_{\mathfrak{n}_a}/\mathcal{I}_{\mathbb{Q}}(V)\mathbb{R}[x]_{\mathfrak{n}_a}.$$

We say that a is a $\mathbb{R}|\mathbb{Q}$-regular point *if the local ring $\mathcal{R}^{\mathbb{R}|\mathbb{Q}}_{V,a}$ is regular of dimension* $\dim(V)$. *Denote by* $\mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(V)$ *the set of $\mathbb{R}|\mathbb{Q}$-regular points of V.*

Remark. Denote by $\mathrm{Reg}(V)$ the set of all regular points of V. If V is $\mathbb{Q}$-algebraic, then $\mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(V)$ is a nonempty Zariski open subset of $\mathrm{Reg}(V)$.

Definition 4 ([4]). *We say that V is* $\mathbb{Q}$-determined *if V is $\mathbb{Q}$-algebraic and*

$$\mathrm{Reg}^{\mathbb{R}|\mathbb{Q}}(V) = \mathrm{Reg}(V).$$

Remark. Let V be a $\mathbb{Q}$-algebraic set. Then V is $\mathbb{Q}$-determined if and only if, for every nonsingular point $a \in V$, the local structure of V is described by polynomials in $\mathcal{I}_{\mathbb{Q}}(V)$ via the Jacobian Criterion.

Main results

Developing new algebraic approximation techniques over $\mathbb{Q}$, we give a positive answer to the previous Open Question in the following cases:

Theorem A ([4, Relative Nash-Tognoli's theorem over $\mathbb{Q}$]). *Let $X \subset \mathbb{R}^n$ be a compact $\mathcal{C}^\infty$ manifold and let $\{X_i\}_i$ be a finite family of $\mathcal{C}^\infty$ hypersurfaces of X in general position. Then there exist $m \geq n$, a nonsingular $\mathbb{Q}$-determined algebraic set $X' \subset \mathbb{R}^m$, nonsingular $\mathbb{Q}$-determined algebraic hypersurfaces $\{X'_i\}_i$ of X' in general position and a diffeomorphism $\varphi : X \to X'$ such that $\varphi(X_i) = X'_i$.*

Corollary A. *If in addition X and all the X_i's in Theorem A are algebraic sets, then $\varphi : X \to X'$ can be chosen to be a Nash diffeomorphism.*

After reduction to the compact case, Theorem A and Corollary A play a crucial role in the proof of our main theorem.

Theorem B ([4, Main theorem]). *Let $V \subset \mathbb{R}^n$ be an algebraic set with isolated singularities. Then there exist $m \geq n$, an algebraic set with isolated singularities $V' \subset \mathbb{R}^m$ and a semialgebraic homeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ such that:*

1. *V' is $\mathbb{Q}$-determined,*
2. *$\phi(V) = V'$ and $\varphi(\mathrm{Reg}(V)) = \mathrm{Reg}(V')$,*
3. *$\phi|_{\mathrm{Reg}(V)}$ is a Nash diffeomorphism.*

Ingredients for the proofs

Key steps for Theorem A and Corollary A:

- Properties of the ring $\mathcal{R}^{\mathbb{R}|\mathbb{Q}}_{V,a}$ to study $\mathbb{Q}$-irreducible components of $\mathbb{Q}$-algebraic sets.
- New relative algebraic approximation techniques over $\mathbb{Q}$ in the spirit of [7] and [1].
- $\mathbb{Q}$-algebraic representatives of bordism classes of smooth functions and $\mathbb{Q}$-determined representatives of $\mathbb{Z}/2\mathbb{Z}$-homological cycles of algebraic sets.
- Nash approximation techniques of smooth maps between Nash manifolds [2, Theorems 1.7 and 1.8].

Key steps for Theorem B:

- Algebraic Alexandroff compactification.
- Lagrange type interpolation to reduce the study to the case in which $\mathrm{Sing}(V) \subset \mathbb{Q}^n$.
- (a) Resolution of singularities, (b) Corollary A, (c) Further approximations over $\mathbb{Q}$ and (d) Our new version over $\mathbb{Q}$ of the real blowing-down lemma of [1].

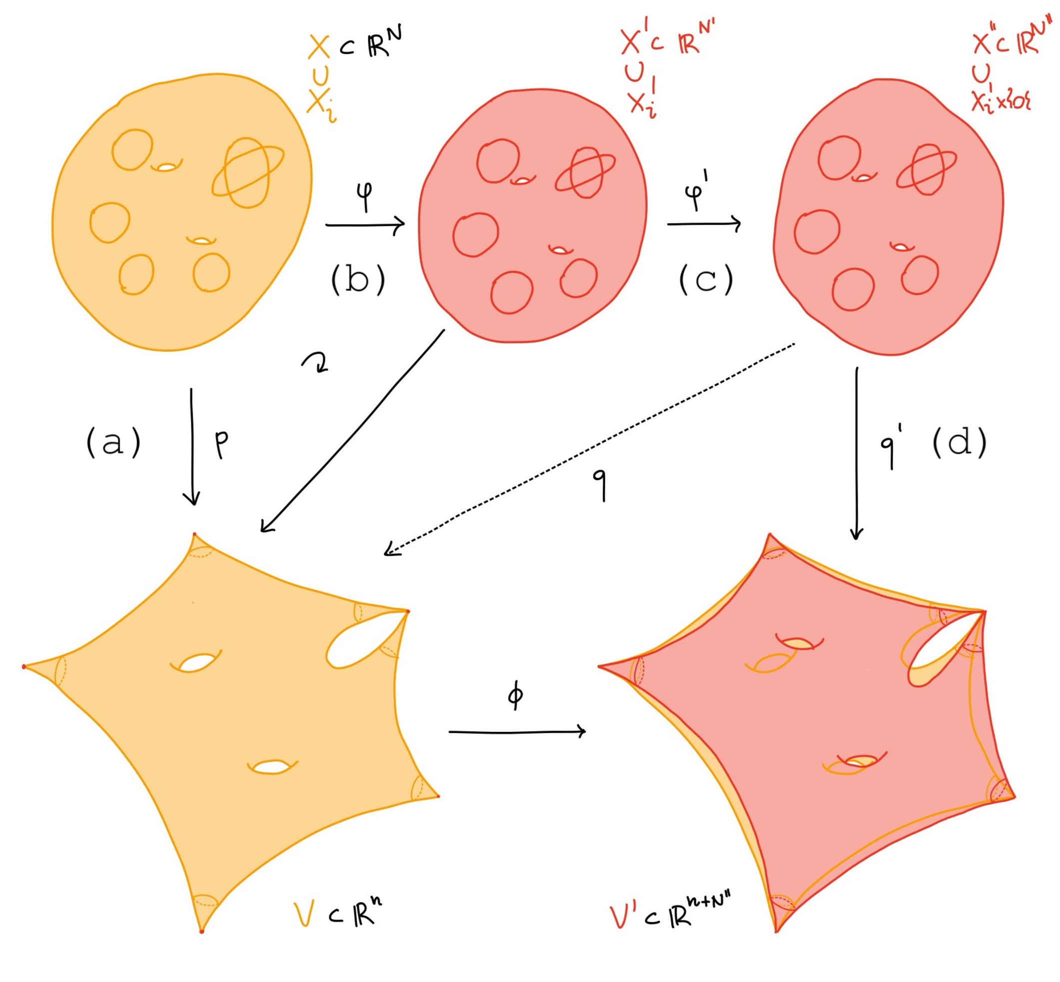

Fig. Sketch of the proof of Theorem B.

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