AN INTRODUCTION TO GEOMETRIC MEASURE THEORY AND AN APPLICATION TO MINIMAL SURFACES (DRAFT DOCUMENT)

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Contents

SOME BASIC NOTATION

If A, B are sets then the symmetric difference between A and B will be denoted by

$$A\Delta B := (A \setminus B) \cup (B \setminus A)\,.$$

We shall tipically work in a metric space X with a metric d, although we will present some notions and results in more general settings. In some chapters however we mainly deal with the Euclidean n- space $\mathbb{R}^n$. Here are the basic notations used in metric spaces throughout these notes. The closed and open balls with centre $x \in X$ and radius r, $0 < r < \infty$, are denoted by

$$B(x,r) = \{y \in X : d(x,y) \leqslant r\}\ ,$$

$$U(x,r) = \{y \in X : d(x,y) < r\}\ .$$

In $\mathbb{R}^n$ we also set

$$B(r) = B(0,r), U(r) = U(0,r), S(x,r) = \partial B(x,r) \text{ and } S(r) = \partial B(0,r);$$

If $B = B(x,r)$ (respectively $B = U(x,r)$) and $\alpha > 0$, we denote $\alpha B = B(x, \alpha\, r)$ (respectively $\alpha B = U(x, \alpha\, r)$). When $\alpha = 5$ we will call $5B$ an *enlargement of* B and we will denote it by $\hat{B}$.

The diameter of a nonempty subset $A \subset X$ is

$$d(A) = \operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}\,.$$

We agree $d(\emptyset) = 0$. If $x \in X$ and A and B are non-empty subsets of X, the distance from x to A and the distance between A and B are, respectively,

$$d(x,A) = \inf\{d(x,y) : y \in A\}\ ,$$

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}\ .$$

For $\epsilon > 0$ the open ϵ-neighbourhood of A is

$$I_\epsilon(A) = \{x \in X : d(x,A) < \epsilon\}\,.$$

If $A \subset \mathbb{R}^n$, then

$$|A| = \mathcal{L}^n(A)$$

where $\mathcal{L}^n$ denote the n-dimensional Lebesgue outer measure.

1. Recalls and complements of measure theory ([AFP, GZ, Mag, R1, SC]).

Motivation: The main goal is to recall and complement some notions and results of measure theory such as: outer measure, measure, signed measures and vector measure with their properties and relationships; measurable functions and their properties; L^p spaces and Riesz representation theorem; convergence of measures.

1.1. **Measures and outer measures, approximation of measures.**

Measures and outer measures.

Let us quickly recall some important notions and results of abstract measure theory (see [GZ]).

Tipically there are two approachs in abstract measure theory: one by using measure, may be more ordinary in the literature, and one by outer measure due to Carathéodory.

Firstly let us introduce the so-called set-theoretic approach where we introduce the notion of "measure" and "measurable set", only assuming that the environment X is a set.

Definition 1.1. Let X denote a set and $\mathcal{P}(X)$ denote the class of all subsets of X.

(i) A set function $\varphi : \mathcal{P}(X) \to [0,\infty]$ is called an outer measure (o.m.) on X if

(OM1)
$$\varphi(\emptyset) = 0\,,$$

(OM2)
$$\varphi(A) \leqslant \varphi(B) \text{ if } A \subset B \quad (\underline{\text{monotonicity}})\,,$$

(OM3)
$$\varphi(\cup_{i=1}^{\infty} A_i) \leqslant \sum_{i=1}^{\infty} \varphi(A_i) \quad \text{for any sequence } (A_i)_i \subset X \quad (\underline{\text{countable subadditivity}})\,.$$

(ii) A set $E \subset X$ is called φ-measurable (with respect to an o.m. φ on X), if
$$\varphi(A) = \varphi(A \cap E) + \varphi(A \setminus E) \quad \forall\, A \subset X\,.$$
The class of φ-measurable sets will be denoted by $\mathcal{M}_\varphi$.

(iii) A σ-algebra $\mathcal{M}$ on X is a (nonempty) class of subsets $\mathcal{M} \subset \mathcal{P}(X)$ satisfying the two following properties:

(σA1)
$$\text{if } E \in \mathcal{M}, \text{ then } X \setminus E \in \mathcal{M}\,;$$

(σA2)
$$\text{for each sequence } (E_i)_i \subset \mathcal{M}, \text{ then } \cup_{i=1}^{\infty} E_i \in \mathcal{M}\,.$$

(iv) A measure μ on X is a set function $\mu : \mathcal{M} \to [0,\infty]$, where $\mathcal{M}$ is a σ-algebra on X, satisfying the following two properties:

(M1)
$$\mu(\emptyset) = 0\,;$$

(M2)
$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \quad (\underline{\text{countable additivity}})$$

for each disjoint sequence $(E_i)_i \subset \mathcal{M}$. The structure composed by the triple $(X, \mathcal{M}, \mu)$, or also the couple $(X, \mathcal{M})$, is called measure space and the sets contained in $\mathcal{M}$ are called measurable sets.

(v) Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is said to be finite, if $\mu(X) < \infty$; it is said to be σ-finite, if there exists a sequence $(X_i)_i \subset \mathcal{M}$ such that $X = \cup_{i=1}^{\infty} X_i$ and $\mu(X_i) < \infty$ for each i.

(v) Let $(X, \mathcal{M}, \mu)$ be a measure space. A point $x \in \mathcal{M}$ is said to be an *atom* if the singleton $\{x\} \in \mathcal{M}$ and $\mu(\{x\}) > 0$. The set of atoms of μ will be denoted by S_μ and μ is said to be *atomic* if $S_\mu \neq \emptyset$. If μ is finite or σ-finite the set of atoms S_μ is at most countable.

(vii) A function $f : X \to \overline{\mathbb{R}} := [-\infty, \infty]$ is called measurable with respect to an o.m. φ (respectively with respect to a measure $\mu : \mathcal{M} \to [0, \infty]$) if $f^{-1}(U)$ is φ-measurable (respectively $f^{-1}(U) \in \mathcal{M}$) for each open set U in $\overline{\mathbb{R}}$.

(viii) A simple function $s : X \to \mathbb{R}$ is one that assumes only a finite number of values. More precisely, s is a simple function if and only if it can be represented as

$$s(x) = \sum_{i=1}^{k} a_i \, \chi_{A_i}(x) \quad \forall\, x \in X \,,$$

with $a_i \in \mathbb{R}$, $A_i \subset X$ $(i = 1, \ldots, k)$, $X = \cup_{i=1}^{k} A_i$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

Based on the ideas of H. Lebesgue, it is well known that a theory of an abstract integration can be carried out on a general measure space $(X, \mathcal{M}, \mu)$ and we refer to [GZ, Chap. 6] or [R1, Chap. 1] for its complete treatment. In particular, let us recall that, given a measurable function $f : X \to [-\infty, \infty]$, it is possible to make a sense to the value *integral of* f with respect to μ, denoted

$$\int_X f \, d\mu \in [-\infty, \infty] \,.$$

When the integral is finite, that is $\int_X f \, d\mu \in (-\infty, \infty)$, f is said to be *integrable* or also *summable.*

Example 1.2. Let $(X, \mathcal{M})$ be a measure space, then we define the following set functions on $\mathcal{M}$, which turns out to be measures, as it can easily be proved.

(i) (counting measure) We define the set function $\# : \mathcal{M} \to [0, \infty]$, $\#(\emptyset) := 0$, $\#(E)$ as the cardinality of E if it is finite, $\#(E) = \infty$ otherwise.

(iI) (Dirac measures) With each $x \in X$ we associate the set function $\delta_x : \mathcal{M} \to [0, \infty]$ defined by $\delta_x(E) := 1$ if $x \in E$, $\delta_x(E) := 0$ otherwise. If $(x_h)_h \subset X$ and $(c_h)_h \subset [0, \infty)$ is a sequence such that the series $\sum_{h=1}^{\infty} c_h$ is convergent, we can define the set function $\sum_{h=1}^{\infty} c_h \, \delta_{x_h} : \mathcal{M} \to [0, \infty]$

$$\left(\sum_{h=1}^{\infty} c_h \, \delta_{x_h} \right) (E) := \sum_{\{h :\, x_h \in E\}} c_h \,.$$

Measures of this kind are called *purely atomic.*

Now, we enrich the environment X, by adding a topology and we require compatibility between topology and measure.

Definition 1.3. Let (X, τ) denote a topological space and denote $\mathcal{B}(X)$ the σ-algebra of Borel sets of X, i.e. the smallest σ-algebra of X which contains the open and closed sets of X.

(i) An o.m. φ on X is called a Borel o.m. if the class of φ-measurable sets $\mathcal{M}_\varphi \supset \mathcal{B}(X)$.

(ii) An o.m. φ on X is called a Borel regular o.m. if it is a Borel o.m. and for each $A \subset X$ there exists $B \in \mathcal{B}(X)$ with $B \supset A$ and $\varphi(A) = \varphi(B)$.

(iii) An o.m. φ on X is called a Radon o.m. if it is a Borel regular o.m. and $\varphi(K) < \infty$ for each compact set $K \subset X$.

(iv) An o.m. φ on a metric space (X, d) is called a Carathéodory o.m. (or also a metric o.m.) if

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) \quad \forall\, A,\, B \subset X \text{ with } d(A, B) > 0\,,$$

where $d(A, B) := \inf\{d(a, b) :\, a \in A, b \in B\}$.

(v) A measure $\mu : \mathcal{M} \to [0, \infty]$ on X is called a Borel measure if $\mathcal{M} = \mathcal{B}(X)$.

(vi) A measure $\mu : \mathcal{M} \to [0, \infty]$ on X is called a Radon measure if it is a Borel measure and $\mu(K) < \infty$ for each compact set $K \subset X$.

(vii) An o.m φ on X(respectively a Borel measure $\mu : \mathcal{B}(X) \to [0, \infty]$) is said to locally finite if for each $x \in X$ there exist an open neiborghood U_x of x such that $\varphi(U_x) < \infty$ (respectively $\mu(U_x) < \infty$).

Remark 1.4. We stress that the notion of Radon o.m. (respectively Radon measure) in a general topological space (X, τ) may actually differ in the current literature from one given in Definition 1.3 (iii) (respectively Definition 1.3 (vi)). Indeed it could be requested that φ (respectively μ) must satisfy to be finite on compact sets and approximation properties (i) and (ii) of Theorem 1.14. The two notions agree on a separable, locally compact metric space (X, d) because of Theorem 1.14.

The following basic properties of outer measures are well known (see [GZ]).

Theorem 1.5. (i) *Let φ be an o.m. on X. Then the class of φ-measurable sets $\mathcal{M}_\varphi$ is a σ-algebra on X and $\varphi : \mathcal{M}_\varphi \to [0, \infty]$ is a measure.*

(ii) *If $\varphi(N) = 0$, then $N \in \mathcal{M}_\varphi$.*

(iii) (Carathéodory's criterion) *Let φ be a Carathéodory o.m. on a metric space (X, d). Then φ is a Borel o.m.*

Remark 1.6. The property of Theorem 1.5 (ii) is characteristic to an outer measure but is not enjoyed by measures. A measure $\mu : \mathcal{M} \to [0, \infty]$ with the property that all subsets of sets of μ-measure zero are measurable, is said to be **complete** and $(X, \mathcal{M}, \mu)$ is called **complete measure space**. Not all measures are complete, but this is not a crucial defect since every measure can easily be completed by enlarging its domain of definition to include all subsets of measure zero, that is by replacing $\mathcal{M}$ with its **completion** denoted $\mathcal{M}^*$ (see [R1, Theorem 1.36] or [GZ, Theorem 4.45]).

Example 1.7. (i) Let $\mathcal{L}^n$ denote the Lebesgue o.m. on $\mathbb{R}^n$. Then $\mathcal{L}^n$ is a Radon measure on $\mathbb{R}^n$. The class $\mathcal{M}_n \equiv \mathcal{M}_{\mathcal{L}^n}$ is called the class of Lebesgue measurable sets.

(ii) Let $s \in [0, \infty)$ and $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure on $\mathbb{R}^n$. Then $\mathcal{H}^s$ is a Borel regular o.m. on $\mathbb{R}^n$, but it is not a Radon o.m. unless $s \geqslant n$. We will deeply study these measures in Chapter III.

(iii) Let A be a non Borel set of $\mathbb{R}^n$ (why does A exists?); let $\varphi : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ be the set function defined as

$$\varphi(E) := \begin{cases} 0 & \text{if } E \subset A \\ +\infty & \text{if } E \setminus A \neq \emptyset \end{cases} .$$

It is easy to see, by the definition, that φ is a Carathéodory outer measure on $\mathbb{R}^n$. However it is not Borel regular because it does not exist a Borel set $B \supset A$ such that $\varphi(B) = \varphi(A)$.

(iv) Let $\varphi: \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ be the set function defined as

$$\varphi(E) := \begin{cases} 0 & \text{if } E = \emptyset \\ 1 & \text{if } E \neq \emptyset \end{cases} .$$

It is easy to see, by the definition, that φ is an o.m. on $\mathbb{R}^n$ and $\mathcal{M}_\varphi = \{\emptyset, \mathbb{R}^n\}$. In particular, it is not a Borel o. m.

By Theorem 1.5 (i) we see that to every o.m. φ on X is associated the measure space $(X, \mathcal{M}_\varphi, \varphi)$

Question: Given a measure space $(X, \mathcal{M}, \mu)$ is there an associated o.m. $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that $\mu^* = \mu$ on $\mathcal{M}$?

There is a simple procedure due to Carathéodory to generate from a measure $\mu : \mathcal{M} \to [0, \infty]$ an outer measure μ^*. Moreover μ^* is also unique (see [GZ, Theorems 4.47 and 4.48]).

Theorem 1.8 (Carathéodory-Hahn extension theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, let*

$$\mu^*(E) := \inf\{\mu(A) : A \supset E, A \in \mathcal{M}\}$$

for each $E \subset X$. Then

(i) *μ^* is an o.m.;*
(ii) *$\mu^*(A) = \mu(A)$ whenever $A \in \mathcal{M}$;*
(iii) *$\mathcal{M} \subset \mathcal{M}_{\mu^*}$;*
(iv) *Let $\mathcal{N}$ be a σ-algebra with $\mathcal{M} \subset \mathcal{N} \subset \mathcal{M}_{\mu^*}$ and suppose that ν is a measure on $\mathcal{N}$ such that $\nu = \mu$ on $\mathcal{M}$. Then $\nu = \mu^*$ on $\mathcal{N}$, provided that μ is σ-finite.*

μ^* is called the o.m. generated by μ.

Let us recall three important results on approximation of measures by open and closed sets. The first result is also contained in [GZ, Theorem 4.17]).

Theorem 1.9 (Approximation of outer measures by open and closed sets)**.** *Let φ be a Borel (respectively a Borel regular) o.m. on a metric space (X, d) and let $B \subset X$ be a Borel set (respectively a φ-measurable set).*

(i) *Suppose that $\varphi(B) < \infty$, then for each $\epsilon > 0$ there exists a closed set $F \subset B$ such that*

$$\varphi(B \setminus F) < \epsilon .$$

(ii) *Suppose*

$$B \subset \cup_{i=1}^{\infty} V_i$$

where each $V_i \subset X$ is an open set with $\varphi(V_i) < \infty$. Then there is an open set $U \supset B$ such that

$$\varphi(U \setminus B) < \epsilon .$$

Remark 1.10. Notice that, if $\varphi(B) = \infty$, then the conclusion of Theorem 1.9 (i) may fail. For instance, consider $X = \mathbb{R}$, $\varphi = \#$, $B = (0, +\infty)$. Then, for each closed set $F \subset (0, +\infty)$, $\#(B \setminus F) = \infty$. The conclusion of Theorem 1.9 (ii) may also fail by means of the same example if the assumptions are dropped.

The first important consequence of Theorem 1.9 is the following

Corollary 1.11. *Let φ be a Borel (respectively a Borel regular) o.m. on a metric space (X,d). Suppose there exists a sequence of open sets $(V_i)_i \subset X$ such that*

$$(\star) \qquad X = \cup_{i=1}^{\infty} V_i \text{ with } \varphi(V_i) < \infty \quad \forall\, i\,.$$

Then for each $B \in \mathcal{B}(X)$ (respectively $B \in \mathcal{M}_\varphi$)

(i) $\varphi(B) = \inf\{\varphi(U) : U \supset B, U \,\text{open}\}$;
(ii) $\varphi(B) = \sup\{\varphi(C) : C \subset B, C \,\text{closed}\}$.

Proof. See, for instance, [SC, Corollary 1.19]. □

The second important consequence of Corollary 1.11 and the Carathéodory-Hahn extension theorem (Theorem 1.8) is the following approximation result for Borel measures.

Corollary 1.12 (Approximation of Borel measures by open and closed sets)**.** *Consider a measure space $(X, \mathcal{B}(X), \mu)$ where X is a metric space and μ is a Borel measure. Suppose that the assumption $(\star)$ of Corollary 1.11 holds replacing φ with μ. Then for each $B \in \mathcal{B}(X)$*

(i) $\mu(B) = \inf\{\mu(U) : U \supset B, U \,\text{open}\}$;
(ii) $\mu(B) = \sup\{\mu(C) : C \subset B, C \,\text{closed}\}$.

Proof. See, for instance, [SC, Corollary 1.20]. □

When (X,d) is a separable, locally compact metric space and φ (respectively μ) is a Radon outer measure (respectively Radon measure) on X, the assumption $(\star)$ is satisfied. Thus the conclusions of Corollary 1.11 (respectively Corollary 1.12) hold. Moreover the approximation from below by means of closed sets can be replaced by compact sets.

Let us recall

Definition 1.13. Let (X,τ) be a topological space.

(i) (X,τ) is said to be *separable* if it has a countable dense subset.
(ii) (X,τ) is said to be *locally compact* if for each $x \in X$ there is an open set $O \ni x$ such that the closure of O, denoted by $\overline{O}$, is compact .

Theorem 1.14 (Approximation of Radon measures on l.c.s. metric spaces)**.** *Let (X,d) be a separable, locally compact metric space and φ (respectively μ) be a Radon outer measure (respectively Radon measure) on X. Then*

(i) *for each $B \subset X$, $\varphi(B) = \inf\{\varphi(U) : U \supset B, U \,\text{open}\}$*
(respectively, for each $B \in \mathcal{B}(X)$, $\mu(B) = \inf\{\mu(U) : U \supset B, U \,\text{open}\}$);
(ii) *for each $B \in \mathcal{M}_\varphi$, $\varphi(B) = \sup\{\varphi(K) : K \subset B, K \,\text{compact}\}$*
(respectively, for each $B \in \mathcal{B}(X)$, $\mu(B) = \sup\{\mu(K) : K \subset B, K \,\text{compact}\}$).

Remark 1.15. An immediate consequence of Theorem 1.14 is that, if two Radon measures on a locally compact metric space (X,d) agree on the class open set, then they have to agree on $\mathcal{P}(X)$.

Before the proof of Theorem 1.14 we need the following topological results, whose the former is well known (see, for instance, [Ro, Proposition 7.6]).

Lemma 1.16. *Let* (X,d) *be a separable metric space and let* $D = \{x_i : i \in \mathbb{N}\} \subset X$ *be dense. Then the family of open sets*

$$\mathcal{U} := \{U(x_i, q) : i \in \mathbb{N}, q \in \mathbb{Q} \cap (0,\infty)\}$$

is a basis for the topology induced on X *by the distance, where* $U(x,r) := \{y \in X : d(x,y) < r\}$ *if* $x \in X$ *and* $r > 0$.

Lemma 1.17. *Let* (X,d) *be a separable, locally compact metric space. Then there exists an increasing sequence of open sets* $(V_i)_i$ *such that*

$$X = \cup_{i=1}^{\infty} V_i, \quad \overline{V_i} \text{ is compact for each } i\,. \tag{1.1}$$

Proof of Lemma 1.17. Recall that, by definition, (X,d) is locally compact if and only if $\forall\, x \in X\, \exists\, r_x > 0$ such that $\overline{U(x,r_x)}$ is compact. Let $D = \{x_i : i \in \mathbb{N}\} \subset X$ be dense. From Lemma 1.16, the family $\mathcal{U}$ is a basis for the topology and let us enumerate $\mathcal{U}$, that is assume that $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$.Thus there exists a set $I(x) \subset \mathbb{N}$ for which

$$U(x,r_x) = \cup_{i \in I(x)} U_i\,.$$

In particular, there exists a choice function $\alpha : X \to \mathbb{N}$ satisfying:

$$x \in U_{\alpha(x)} \text{ and } U_{\alpha(x)} \subset U(x,r_x) \subset \overline{U(x,r_x)}\,. \tag{1.2}$$

Let $J := \alpha(X) \subset \mathbb{N}$ and

$$V_i := \cup_{j \in (J \cap \{1,\dots,i\})} U_j \text{ if } i \in \mathbb{N}\,.$$

Then, by (1.2), (1.1) follows. □

Proof of Theorem 1.14. Let us first notice that, without loss of generality, we can assume that $B \in \mathcal{B}(X)$. Indeed, if not, since φ is a Borel regular o.m., we can replace B by a Borel set $\tilde{B} \supset B$ and $\varphi(\tilde{B}) = \varphi(B)$. By Lemma 1.17, ($\star$) of Corollary 1.11 is satisfied. Thus claim (i) follows at once from Corollary 1.11 (i) (respectively from Corollary 1.12 (i)). Let us now prove (ii) for a given φ Radon o.m. and $B \in \mathcal{M}_\varphi$. Since each compact set is also closed, from Corollary 1.11 (ii), it follows that

$$\varphi(B) = \sup\{\varphi(C) : C \subset B, C \,\text{closed}\} \geqslant \sup\{\varphi(K) : K \subset B, K \,\text{compact}\}\,.$$

Thus we have only to prove that

$$\varphi(B) = \sup\{\varphi(C) : C \subset B, C \,\text{closed}\} \leqslant \sup\{\varphi(K) : K \subset B, K \,\text{compact}\} \tag{1.3}$$

Let $C \subset B$ be a closed set, let $(V_i)_i$ be the sequence of open sets in (1.1) and let

$$K_i := C \cap (\cup_{j=1}^{i} \overline{V_j})\,.$$

Then $(K_i)_i$ is an increasing sequence of compact sets such that

$$C := \cup_{i=1}^{\infty} K_i\,.$$

Observe now that

$$\varphi(K_i) \leqslant \sup\{\varphi(K) : K \subset B, K \,\text{compact}\} \text{ for each } i\,, \tag{1.4}$$

and, by the continuity of o.m. φ on increasing sequences of measurable sets,

$$\lim_{i \to \infty} \varphi(K_i) = \varphi(C)\,. \tag{1.5}$$

By (1.4) and (1.5), (1.3) follows. □

1.2. **Convergence and approximation of measurable functions: Severini-Egoroff's and Lusin's theorems.**

Theorem 1.18 (Severini-Egoroff). *Let $(X, \mathcal{M}, \mu)$ be a measure space with μ finite. Suppose $f_h : X \to \overline{\mathbb{R}}$ $(h = 1, 2, \dots)$ and $f : X \to \overline{\mathbb{R}}$ are measurable functions that are finite μ-a.e.on X. Also, suppose that $(f_h)_h$ converges pointwise μ-a.e. to f. Then for each $\epsilon > 0$ there exists a set $A \in \mathcal{M}$ such that $\mu(X \setminus A) < \epsilon$ and $f_h \to f$ uniformly on A, that is*

$$\sup_{x \in A} |f_h(x) - f(x)| \to 0 \text{ as } h \to \infty .$$

.

Proof. See [GZ, Theorem 5.15]. □

Remark 1.19. The hypothesis that $\mu(X) < \infty$ is essential in Severini-Egoroff's theorem. Consider the case of Lebesgue measure $\mathcal{L}^1$ on $\mathbb{R}$ and define a sequence of functions by

$$f_h = \chi_{[h,\infty)} ,$$

for each positive integer h. Then, $\lim_{h\to\infty} f_h(x) = 0$ for each $x \in \mathbb{R}$, but $(f_h)_h$ does not converge uniformly to 0 on any set A whose complement $\mathbb{R} \setminus A$ has finite Lebesgue measure. Indeed, it would follow that $\mathbb{R} \setminus A$ does not contain any half-line $[h, \infty)$; that is, for each h, there would exist $x \in [h, \infty) \cap A$ with$f_h(x) = 1$, thus showing that $(f_h)_h$ does not converge uniformly to 0 on A.

Theorem 1.20 (Approximation by simple functions). *Let $(X, \mathcal{M})$ be a measure space and let $f : X \to [0, +\infty]$ be a measurable function. Then there exists a sequence of measurable simple functions $s_h : X \to [0, +\infty)$ $(h = 1, 2, \dots)$ satisfying the properties:*

(i) $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_h \leqslant \ldots \leqslant f$;
(ii) $\lim_{h\to\infty} s_h(x) = f(x) \quad \forall\, x \in X$.

In particular, if $\int_X f\, d\mu < \infty$, then

$$\int_X |f - s_h|\, d\mu \to 0 .$$

Proof. See [GZ, Theorem 5.25]. □

Let us now introduce two spaces of continuous functions which play an important role in measure theory.

Definition 1.21. Let (X, τ) be a topological space.

(i)

$$\mathbf{C}^0_c(X) := \{f : X \to \mathbb{R} : \text{ f is continuous and } \mathrm{spt}(f) \text{ is compact in } (X, \tau)\}$$

where

$$\mathrm{spt}(f) := \text{closure}\, \{x \in X : f(x) \neq 0\} . \tag{1.6}$$

(ii)

$$\mathbf{C}^0_b(X) := \{f : X \to \mathbb{R} : \text{ f is continuous and bounded}\}$$

Remark 1.22. Observe that, if a topological space (X, τ) is not locally compact, the space $\mathbf{C}^0_c(X)$ could turn out to be meaningless, that is $\mathbf{C}^0_c(X) = \{0\}$.Indeed

Exercise: Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space. Then $\mathbf{C}^0_c(X) = \{0\}$.

Lusin's theorem 1.23 (1912, form on locally compact metric spaces)**.** *Let μ be a Radon outer measure on a locally compact, separable metric space X . Let $f : X \to \overline{\mathbb{R}}$ be a μ-measurable function such that there exists a Borel set $A \subset X$ with*

$$\mu(A) < \infty,\ f(x) = 0 \quad \forall\, x \in X \setminus A \text{ and } |f(x)| < \infty \quad \mu - a.e.\ x \in X\,.$$

Then, for each $\epsilon > 0$, there exists $g \in \mathbf{C}^0_c(X)$ such that

$$\mu\left(\{x \in X :\ f(x) \neq g(x)\}\right) < \epsilon\,.$$

Moreover g can be chosen such that

$$\sup_{x\in X} |g(x)| \leqslant \sup_{x\in X} |f(x)|\,.$$

Proof. See [R1, Theorem 2.23]. □

A consequence of Lusin's theorem is the following useful approximation of measurable functions by means of Borel functions.

Corollary 1.24. *Let μ be a Radon outer measure on a locally compact, separable metric space X and let $f : X \to \mathbb{R}$ be a μ-measurable function. Then there exist a Borel function $g : X \to \mathbb{R}$ such that $f = g\ \mu$-a.e. on X.*

Proof. See, for instance, [Fe, 2.3.6]. □

1.3. **Absolutely continuous and singular measures. Radon-Nikodym and Lebesgue decomposition theorems.** Firstly, let us introduce some definitions and preliminary results.

Definition 1.25. Let $(X\,, \mathcal{M})$ be a measure space and let $\mu\,, \nu :\ \mathcal{M} \to [0, \infty]$ be two measures.

(i) The measure ν is said to be absolutely continuous with respect to the measure μ, written $\nu << \mu$, if it holds that

$$\mu(E) = 0 \Rightarrow \nu(E) = 0\,.$$

(ii) The measures ν and μ are said to be mutually singular, written $\mu \perp \nu$, if there exists a measurable set E such that

$$\nu(E) = \mu(X \setminus E) = 0\,.$$

The following result justifies why the word "continuity" is used in this context.

Theorem 1.26. *Let ν be a finite measure and μ a measure on a measure space $(X, \mathcal{M})$. Then the following are equivalent:*

(i) $\nu << \mu$;

(ii) $\lim_{\mu(A)\to 0} \nu(A) = 0$, *that is, for every $\varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0$ such that $\nu(E) < \varepsilon$ whenever $\mu(E) < \delta$.*

Proof. See [GZ, Theorem 6.33]. □

Let us now introduce two fundamental results of measure theory: the Radon-Nikodym and Lebesgue's decomposition theorems.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $w : X \to [0, \infty]$ be measurable. Then it is easy to see that μ_w is a measure, absolutely continuous w.r.t. μ. The very remarkable fact, content of the Radon-Nikodym theorem, is that essentially each measure ν, absolutely continuous w.r.t. μ is of this form.

Theorem 1.27 (Radon-Nikodym)**.** *Let ν and μ be two measures on $(X, \mathcal{M})$. Suppose that*

(i) *ν and μ are σ-finite, that is, there exists a sequence $(X_i)_i \subset \mathcal{M}$ such that $X = \cup_{i=1}^{\infty} X_i$ and*

$$\nu(X_i) < \infty \text{ and } \mu(X_i) < \infty \quad \text{for each } i\,.$$

(ii) *$\nu << \mu$.*

Then there exists a measurable function $w : X \to [0, \infty]$ such that $\nu = \mu_w$ on $\mathcal{M}$, that is,

$$\text{(RN)} \qquad \nu(E) = \mu_w(E) := \int_E w\, d\mu \quad \forall E \in \mathcal{M}\,.$$

Moreover the function w in (RN) *is μ-a.e. unique.*

Definition 1.28. The function w in (RN) is called the Radon-Nikodym derivative of ν with respect to μ and denoted by $w = \dfrac{d\nu}{d\mu}$.

Remark 1.29. Because ν is σ-finite, then w is also σ-integrable with respect to μ, that is,

$$(\sigma\text{I}) \qquad 0 \leqslant \int_{X_i} w\, d\mu < \infty \quad \forall i\,,$$

where $(X_i)_i$ is the sequence in statement (i).

Proof. See See [GZ, Theorem 6.38] and also [SC, Theorem 1.30]. □

Remark 1.30. We can actually weaken the assumptions of the Radon-Nikodym theorem. Indeed it is sufficient to require that only μ is σ-finite in order that (RN) holds (see, for instance,[Ro, Theorem 23, Chap. 11]). When μ is not σ-finite, the Radon- Nikodym theorem fails (see Exercise I.8).

For Radon measures on locally compact, separable metric spaces, the Radon-Nikodym theorem has the following simpler and stronger version.

Theorem 1.31 (Radon-Nikodym's theorem for Radon measures)**.** *If X is supposed to be a locally compact, separable metric space and ν and μ are Radon measures on X with $\nu << \mu$, then* (RN) *holds and the Radon-Nikodym derivative $w := \frac{d\nu}{d\mu}$ is locally integrable on X, i.e. $w \in L^1_{loc}(X, \mu)$, where*

$$L^1_{loc}(X, \mu) := \left\{ f : X \to \overline{\mathbb{R}} : f \text{ is measurable}, \int_K |f|\, d\mu < \infty \text{ for each compact } K \subset X \right\}.$$

Proof. See [SC, Theorem 1.35]. □

Historical notes: The first version of Radon-Nikodym's theorem is due to H. Lebesgue ([Le]) and to G. Vitali ([Vitali]) when $X = \mathbb{R}$ in 1904. Radon extended the result when $X = \mathbb{R}^n$ ([Ra]) in 1913. Eventually Nikodym ([Ni]) extended the result to the abstract setting in 1930.

A consequence of the Radon-Nikodym theorem is the following

Lebesgue decomposition theorem 1.32. *Let ν and μ be σ-finite measures on a measure space $(X, \mathcal{M})$. Then there is a decomposition of ν such that*

$$\nu = \nu_{ac} + \nu_s \,,$$

where ν_{ac} and ν_s are still measures on $(X, \mathcal{M})$ with $\nu_{ac} << \mu$ and $\nu_s \perp \mu$. The decomposition is unique.

Proof. See [GZ, Theorem 6.39] and also [SC, Theorem 1.36]. □

Exercise I.10 Let consider $\mu = \mathcal{L}^1$, $\nu = \delta_0$ as measures on the σ- algebra $\mathcal{M}_1$ of Lebesgue measurable sets in $\mathbb{R}$,where δ_0 denotes the Dirac measure at 0, that is, $\delta_0(E) := \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E\,. \end{cases}$. Prove that the Lebesgue decomposition of ν with respect to μ, $\nu = \nu_{ac} + \nu_s$, is given by $\nu_{ac} \equiv 0$ and $\nu_s = \nu$.

1.4. **Signed vector measures.** Lebesgue decomposition theorem still holds for a more general class of measures. Namely for set functions $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ which still verify basic properties of countable additivity.

Definition 1.33 (Signed measures)**.** Let $(X, \mathcal{M})$ be a measure space.

(i) An extended real valued set function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ is a *signed measure* if it satisfies the following three properties:

(SM1) ν assumes at most one of the values $+\infty$, $-\infty$;

(SM2) $\nu(\emptyset) = 0$;

(SM3) For each sequence of disjoint sets $(E_i)_i \subset \mathcal{M}$, it holds that

$$\nu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$$

where the series on the right either converges absolutely or diverges to $-\infty$ or $+\infty$.

(ii) A signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ is said to be *absolutely continuous with respect to* $\mu : \mathcal{M} \to [0, \infty]$, written $\nu << \mu$, if $\nu(E) = 0$ whenever $\mu(E) = 0$.

(iii) Two signed measure ν, $\mu : \mathcal{M} \to \overline{\mathbb{R}}$ are said to be *mutually singular* , written $\nu \perp \mu$, if there is $E \in \mathcal{M}$ such that $\nu(E) = \mu(X \setminus E) = 0$.

(iv) A signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ is said to be *finite* (respectively σ*-finite*) if $|\nu(X)| < \infty$ (respectively there exists a sequence $(X_i)_i \subset \mathcal{M}$ such that $X = \cup_{i=1}^{\infty} X_i$ and $|\nu(X_i)| < \infty$ for each i).

(v) A real valued signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}$, that is, if $\nu(E) \in \mathbb{R}$ for each $E \in \mathcal{M}$, is called a *real measure.*

Example 1.34 (Examples of signed measures)**.** Let us introduce below two remarkable examples of signed measures on a given measure space $(X, \mathcal{M})$.

(i) Let $\mu : \mathcal{M} \to [0, \infty]$ be a measure and let $f : X \to \overline{\mathbb{R}}$ be a measurable function. Suppose at least one of $f^+ := f \vee 0$ or $f^- := (-f) \vee 0$ is integrable, and let $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ denote the extended real-valued function on $\mathcal{M}$ defined by

$$\nu(E) := \int_E f \, d\mu \quad \forall E \in \mathcal{M}.$$

Then is easy to see that ν is a signed measure and $\nu << \mu$. If both f^+ and f^- are integrable, or, equivalently, $|f|$ is integrable, then ν is a real measure.

(ii) Let μ_1, $\mu_2 : \mathcal{M} \to [0, \infty]$ be measures and assume that al least one of them is finite. Let $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ denote the extended real-valued function on $\mathcal{M}$ defined by

$$\nu(E) := \mu_1(E) - \mu_2(E) \quad \forall E \in \mathcal{M}.$$

Then is easy to see that ν is a signed measure. If both μ_1 and μ_2 are finite, then ν is a real measure.

Remark 1.35. Observe that a measure is a signed measure. In some contexts we will emphasize that a measure μ is not a signed measure by saying that it is a positive measure. Notice also that a signed (or also real) measure ν is not an increasing set function.

Exercise: A signed measure ν is a real measure if and only if it is finite, that is $|\nu(X)| < \infty$.

Theorem 1.36 (Lebesgue decomposition theorem for signed measures)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space with μ σ-finite, and $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ be a σ-finite signed measure . Then there are two signed measures ν_{ac}, $\nu_s : \mathcal{M} \to \overline{\mathbb{R}}$ such that*

$$\nu_{ac} << \mu, \quad \nu_s \perp \mu, \quad \nu = \nu_{ac} + \nu_s, \tag{LD}$$

and there exists a measurable function $w : X \to \overline{\mathbb{R}}$ such that either w_+ or w_- is integrable with respect to μ such that

$$\nu_{ac}(E) = \int_E w \, d\mu \quad \forall\, E \in \mathcal{M}. \tag{RN}$$

Moreover both decomposition (LD) *and representation* (RN) *are unique.*

Proof. See [F, Theorem 3.8]. □

Remark 1.37. Notice that the sum of signed measures $\nu_{ac} + \nu_s$ is well defined in (LD) since ν_{ac} and ν_s are mutually singular.

Remark 1.38. Suppose ν is a real measure (observe that ν is also σ-finite), that is $\nu : \mathcal{M} \to \mathbb{R}$, and $\nu << \mu$ with μ a given σ-finite positive measure on $\mathcal{M}$. Applying Theorem 1.36, we have that

$$\nu(E) = \nu_{ac}(E) = \int_E w \, d\mu \quad \forall\, E \in \mathcal{M},$$

and $w : X \to \overline{\mathbb{R}}$ is now an integrable function on X with respect to μ, that is $\int_X |w| \, d\mu < \infty$. Indeed, since $\nu(E) \in \mathbb{R}$ for each $E \in \mathcal{M}$, it is easy to see that w must be integrable.

We recommend [GZ, Section 6.5] and [F, Chap. 6] for a complete treatment concerning signed measures. However we point out that signed measures in Example 1.34 are really the only examples: every signed measure can be represented in either of these two forms.

An other important tool in GMT will turn out to be the notion of *signed vector measure*, which is an extension of the one of signed measure.

Definition 1.39 (Vector signed measures)**.** Let $(X, \mathcal{M})$ be a measure space.

(i) A vector set function $\nu = (\nu^{(1)}, \ldots, \nu^{(m)}) : \mathcal{M} \to \overline{\mathbb{R}}^m$ is a *vector signed measure* if its components $\nu^{(i)} : \mathcal{M} \to \overline{\mathbb{R}}$ $(i = 1, \ldots, m)$ are signed measures (according to Definition 1.33).

(ii) A vector signed measure $\nu : \mathcal{M} \to \overline{\mathbb{R}}^m$ is a *vector measure* if it is $\mathbb{R}^m$-valued vector measure, that is $\nu : \mathcal{M} \to \mathbb{R}^m$.

(iii) If ν is a signed vector measure, we define its *total variation* $|\nu| : \mathcal{M} \to [0, \infty]$ as follows:

$$|\nu|(E) = \sup \left\{ \sum_{h=1}^{\infty} |\nu(E_h)| : (E_h)_h \subset \mathcal{M} \text{ pairwise disjoint}, E = \cup_{h=1}^{\infty} E_h \right\} ,$$

where

$$|v| := \begin{cases} |v|_{\mathbb{R}^m} & \text{if } v \in \mathbb{R}^m \\ \infty & \text{if } v \in \overline{\mathbb{R}}^m \setminus \mathbb{R}^m \end{cases} .$$

(iv) If ν is a real measure, that is $\nu : \mathcal{M} \to \mathbb{R}$, we define its *positive* and *negative parts* respectively as follows:

$$\nu^+ = \frac{|\nu| + \nu}{2} \text{ and } \nu^- = \frac{|\nu| - \nu}{2}$$

Notation: In the following, we will say *countable partition of a set* E a pairwise disjoint sequence of sets $(E_h)_h$ such that $\cup_{h=1}^{\infty} E_h = E$.

Remark 1.40. Observe that, according to Definition 1.33 (i), a $\overline{\mathbb{R}}^m$-valued signed vector measure $\nu = (\nu^{(1)}, \ldots, \nu^{(m)})$ satisfies the following two properties:

(SVM1) $\nu(\emptyset) = 0 := (0, \ldots, 0) \in \overline{\mathbb{R}}^m$;

(SVM2) For each sequence of disjoint sets $(E_h)_h \subset \mathcal{M}$, it holds that

$$\nu(\cup_{h=1}^{\infty} E_h) = \sum_{h=1}^{\infty} \nu(E_h) := \left(\sum_{h=1}^{\infty} \nu^{(1)}(E_h), \ldots, \sum_{h=1}^{\infty} \nu^{(m)}(E_h) \right) ,$$

where the series on the right-hand side $\sum_{h=1}^{\infty} \nu^{(i)}(E_h)$ $(i = 1, \ldots, m)$ either converges absolutely or diverges to $-\infty$ or $+\infty$.

Notice also that, if ν is $\mathbb{R}^m$-valued vector measure, then the absolute convergence in the series in (SVM2) is a requirement on the set function ν: in fact the sum of the series cannot depend on the order of its terms, as the union does not. Observe also that, when $m = 1$, the notion of signed vector measure (respectively vector measure) agrees with the one of signed measure (respectively real measure). Eventually notice that a $\mathbb{R}^m$-valued set function $\nu = (\nu_1, \ldots, \nu_m) : \mathcal{M} \to \mathbb{R}^m$ is a vector measure if and only if $\nu_i : \mathcal{M} \to \mathbb{R}$ $(i = 1, \ldots, m)$ is a real measure.

Remark 1.41. The introduction of the notion of total variation solves the problem of finding a positive measure μ which dominates a given signed vector measure ν on $\mathcal{M}$ in the sense that $|\nu(E)| \leqslant \mu(E)$ for each $E \in \mathcal{M}$, looking for keeping μ as small as we can. Every solution to this problem (if there is one at all) must satisfy

$$\mu(E) = \sum_{h=1}^{\infty} \mu(E_h) \geqslant \sum_{h=1}^{\infty} |\nu(E_h)|$$

for each partition $(E_h)_h$ of any set $E \in \mathcal{M}$, so that $\mu(E)$ is at least equal to quantity $|\nu|(E)$. This suggest the reason of total variation's definition like in Definition 1.39 (iii).

Let us show that the total variation of a signed vector measure (respectively vector measure) is a positive measure (respectively positive finite measure).

Theorem 1.42. (i) *Let ν be a signed vector measure on $(X, \mathcal{M})$. Then its total variation $|\nu|$ is a positive measure.*
(ii) *If ν is a vector measure, then $|\nu|$ is a positive finite measure, that is $|\nu|(X) < \infty$.*

Proof. (i) We have to prove that

$$|\nu|(\emptyset) = 0 \tag{1.7}$$

and

$$|\nu| \text{ is countable additive .} \tag{1.8}$$

It is trivial, by definition, that (1.7) holds. Let us show (1.8).

Let us firstly observe that $|\nu| : \mathcal{M} \to [0, \infty]$ is increasing , that is

$$|\nu|(E) \leqslant |\nu|(F) \quad \text{if } E \subset F\,. \tag{1.9}$$

Indeed let $(E_h)_h \subset \mathcal{M}$ be a partition of E, then the family of sets $\{E_h : h\} \cup \{F \setminus E\}$ is a countable partition of F. Thus

$$\sum_{h=1}^{\infty} |\nu(E_h)| \leqslant \sum_{h=1}^{\infty} |\nu(E_h)| + |\nu(F \setminus E)| \leqslant |\nu|(F)\,.$$

Then, taking the supremum on the partitions of E in the previous inequality, we get (1.9). Let us now show that

$$|\nu| \text{ is countably subadditive} \tag{1.10}$$

and

$$|\nu| \text{ is additive .} \tag{1.11}$$

Observe that, from (1.9), (1.10) and (1.11), (1.8) follows. Indeed let $(E_h)_h \subset \mathcal{M}$ be pairwise disjoint. Then by countable subadditivity (1.10),

$$|\nu|(\cup_{h=1}^{\infty} E_h) \leqslant \sum_{h=1}^{\infty} |\nu|(E_h)\,. \tag{1.12}$$

On the other hand, by (1.9) and (1.11),

$$(1.13) \qquad |\nu|(\cup_{h=1}^{\infty} E_h) \geqslant |\nu|(\cup_{h=1}^{m} E_h) = \sum_{h=1}^{m} |\nu|(E_h) \quad \forall m \in \mathbb{N}\,.$$

Thus, by (1.12) and (1.14), (1.8) follows. Let us now prove (1.10). Let E, $(E_h)_h$ be in $\mathcal{M}$ such that $E \subset \cup_{h=1}^{\infty} E_h$. Let us define a pairwise disjoint sequence $(E'_h)_h \subset \mathcal{M}$ in the following way $E'_0 := E_0$ and $E'_h := E_h \setminus \cup_{i=1}^{h-1} E_i$ if $h \geqslant 1$. Let $(F_i)_i$ be a partition of E, since $(E'_h \cap F_i)_h$ is a partition of F_i for fixed i, by countable additivity (SVM2) and (1.9), we can infer

$$\begin{aligned}\sum_{i=1}^{\infty} |\nu(F_i)| &= \sum_{i=1}^{\infty} \left|\sum_{h=1}^{\infty} \nu(E'_h \cap F_i)\right| \leqslant \sum_{i=1}^{\infty}\sum_{h=1}^{\infty} |\nu(E'_h \cap F_i)| \\ &= \sum_{h=1}^{\infty}\sum_{i=1}^{\infty} |\nu(E'_h \cap F_i)| \leqslant \sum_{h=1}^{\infty} |\nu|(E'_h \cap E) \leqslant \sum_{h=1}^{\infty} |\nu|(E_h)\,.\end{aligned}$$

Taking the supremum on the partitions of E in the previous inequality, it follows that

$$|\nu|(E) \leqslant \sum_{h=1}^{\infty} |\nu|(E_h)$$

and (1.10) follows. Let us now prove (1.11). Let E, $F \in \mathcal{M}$ be disjoint. If at least one between $|\nu|(E)$ and$|\nu|(F)$ is ∞, then, by (1.9), it is immediate that

$$|\nu|(E \cup F) = \infty = |\nu|(E) + |\nu|(F)\,.$$

Thus, without loss of generality, we can assume that both $|\nu|(E)$ and$|\nu|(F)$ are finite. By definition, for each $\epsilon > 0$, there exist a partition $(E_h)_h \subset \mathcal{M}$ of E and one $(F_h)_h \subset \mathcal{M}$ of F such that

$$|\nu|(E) \leqslant \sum_{h=1}^{\infty} |\nu|(E_h) + \epsilon, \quad |\nu|(F) \leqslant \sum_{h=1}^{\infty} |\nu|(F_h) + \epsilon\,.$$

Observe now that the family of sets $\{E_h : h \in \mathbb{N}\} \cup \{F_h : h \in \mathbb{N}\}$ is a countable partition of $E \cup F$. Then, by the previous inequality, it follows that, for each $\epsilon > 0$,

$$|\nu|(E) + |\nu|(F) - 2\epsilon \leqslant \sum_{h=1}^{\infty} |\nu|(E_h) + \sum_{h=1}^{\infty} |\nu|(F_h) \leqslant |\nu|(E \cup F)\,.$$

By countable subadditivity (1.10) and the previous inequality, (1.11) follows.
(ii) It is sufficient to assume that ν is a real measure, that is $m = 1$ and $\nu : \mathcal{M} \to \mathbb{R}$. The $\mathbb{R}^m$-valued case being an easy consequence of the following estimate

$$|\nu|(E) \leqslant \sum_{i=1}^{m} |\nu_i|(E) \quad \forall E \in \mathcal{M}\,,$$

if $\nu = (\nu_1, \ldots, \nu_m)$. Suppose that for some $E \in \mathcal{M}$ has $|\nu|(E) = \infty$. Let us then prove there exist two disjoint sets A, $B \in \mathcal{M}$ such that

$$(1.14) \qquad E = A \cup B,\ |\nu(A)| > 1,\ |\nu(B)| > 1,\ \text{ either } |\nu|(A) = \infty \text{ or } |\nu|(B) = \infty.$$

By definition, there is a partition $(E_h)_h$ of E such that

$$\sum_{h=1}^{m} |\nu(E_h)| > 2(|\nu(E)| + 1)\,. \tag{1.15}$$

Let $I := \{1 \leqslant h \leqslant m : \nu(E_h) > 0\}$ and $J := \{1 \leqslant h \leqslant m : \nu(E_h) < 0\}$. Since, by the additivity

$$\sum_{h=1}^{m} |\nu(E_h)| = \sum_{h\in I} \nu(E_h) - \sum_{h\in J} \nu(E_h) = \nu(\cup_{h\in I} E_h) - \nu(\cup_{h\in J} E_h)\,,$$

by (1.15), we can infer that either $|\nu(\cup_{h\in I} E_h)| = \nu(\cup_{h\in I} E_h) > (|\nu(E)| + 1)$ or $|\nu(\cup_{h\in J} E_h)| = -\nu(\cup_{h\in J} E_h) > (|\nu(E)| + 1)$. Let A denote one between sets $\cup_{h\in I} E_h$ and $\cup_{h\in J} E_h$ such that $|\nu(A)| > (|\nu(E)| + 1)$ and let $B := E \setminus A$. Then

$$|\nu(B)| = |\nu(E) - \nu(A)| \geqslant |\nu(A)| - |\nu(E)| > 1\,.$$

By the additivity of $|\nu|$, it is clear that either $|\nu|(A) = \infty$ or $|\nu|(B) = \infty$. Therefore (1.14) follows. Now if $|\nu|(X) = \infty$, then we can apply (1.14) with $E = X$ and split X into two sets A_1 and B_1 with $|\nu(A_1)| > 1$ and $|\nu|(B_1) = \infty$. Split B_1 into two sets A_2 and B_2 with $|\nu(A_2)| > 1$ and $|\nu|(B_2) = \infty$. Continuing in this way, we get a countably infinite disjoint family of sets $(A_h)_h$ with $|\nu(A_h)| > 1$ for each h. The countable additivity of ν implies that

$$\nu(\cup_{h=1}^{\infty} A_h) = \sum_{h=1}^{\infty} \nu(A_h)\,.$$

But this series cannot converge since $\nu(A_h)$ does not tend to 0 as $h \to \infty$. This contradiction shows that $|\nu|(X) < \infty$. □

Remark 1.43. The above theorem shows that for any real measure ν, its positive and negative part are positive finite measures, hence the decomposition $\nu = \nu^+ - \nu^-$ holds; it is known as the *Jordan decomposition* of ν. We point out that a Jordan decomposition still hold for signed measures, by means of a suitable notion of positive and negative parts for a signed measure (see [GZ, Theorem 6.31] or [F, Theorem 3.4]).

Corollary 1.44. *Let $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ be a signed measure. Then ν is σ- finite if and only if so does its total variation $|\nu| : \mathcal{M} \to [0, \infty]$.*

Proof. If $|\nu|$ is σ-finite, since

$$|\nu(E)| \leqslant |\nu|(E) \quad \forall E \in \mathcal{M}\,,$$

according to Definition 1.33 (iv), ν is also σ-finite. Suppose that ν is σ- finite, that is there is a disjoint sequence $(X_k)_k \subset \mathcal{M}$ such that $|\nu(X_k)| < \infty$ for each k. For given k, let us define the set function $\nu_k : \mathcal{M} \to \overline{\mathbb{R}}$ defined by $\nu_k(E) := \nu(E \cap X_k)$. Notice that ν_k is a real measure. Thus, from Theorem 1.42 (ii), its total variation $|\nu_k|$ is a positive finite measure, that is $|\nu_k|(X) < \infty$. Let us now prove that

$$|\nu_k|(X) = |\nu|(X_k) \tag{1.16}$$

from which it will follow that ν is σ-finite and the proof is accomplished. By definition

$$
(1.17) \quad \begin{aligned} |\nu_k|(X) &= \sup\left\{\sum_{h=1}^{\infty}|\nu(E_h\cap X_k)| : (E_h)_h\subset\mathcal{M} \text{ partition of } X\right\} \\ &\leqslant \sup\left\{\sum_{h=1}^{\infty}|\nu(F_h| : (F_h)_h\subset\mathcal{M} \text{ partition of } X_k\right\} = |\nu|(X_k)\,. \end{aligned}
$$

Let $(F_h)_h\subset\mathcal{M}$ be a partition of X_k and define the partition of X as $E_1 := X\setminus X_k$, $E_h := F_{h-1}$ if $h\geqslant 2$.Then it trivial that

$$
(1.18) \qquad \sum_{h=1}^{\infty}|\nu(F_h)| = \sum_{h=1}^{\infty}|\nu(E_h\cap X_k)| = \sum_{h=1}^{\infty}|\nu_k(E_h)| \leqslant |\nu_k|(X)\,.
$$

Therefore, by (1.17) and (1.18), (1.16) follows. □

Remark 1.45. It is immediate to check that $\mathbb{R}^m$-valued vector measures can be added and multiplied by real numbers, hence they form a real vector space; moreover, an easy consequence of Theorem 1.42 is that the total variation is a norm on the space of measures, which turns out to be a Banach space . If X is a locally compact separable metric space, it will be identified with the dual of a space of continuous functions and this will give the completeness in another way (see Corollary 1.78 and Theorem 1.83).

Example 1.46. According to the notation for positive measures (see (RN)), given a measure space $(X,\mathcal{M},\mu)$ and a vector function $w=(w_1,\dots,w_m) : X\to\overline{\mathbb{R}}^m$, with each $w_i : X\to\overline{\mathbb{R}}$ $(i=1,\dots,m)$ measurable functions such that either $w_{i,+}$ or $w_{i,-}$ is integrable. Let us define the vector set function $\mu_w : \mathcal{M}\to\overline{\mathbb{R}}^m$ defined as follows

$$
(1.19) \qquad \mu_w(E) = \int_E w\,d\mu := \left(\int_E w_1\,d\mu,\dots,\int_E w_m\,d\mu\right) \quad E\in\mathcal{M}\,.
$$

Then it is easy to see that μ_w is a signed vector measure and its total variation is computed in the following proposition.

Proposition 1.47. *Let $(X,\mathcal{M},\mu)$ be a measure space and let $w=(w_1,\dots,w_m) : X\to\overline{\mathbb{R}}^m$, with each $w_i : X\to\overline{\mathbb{R}}$ $(i=1,\dots,m)$ measurable functions such that either $w_{i,+}$ or $w_{i,-}$ is integrable. Consider the vector signed measure μ_w in* (1.19)*.*

Then

$$
(1.20) \qquad |\mu_w|(E) = \int_E |w|\,d\mu \quad \forall E\in\mathcal{M}\,.
$$

Proof. It is easy to see that, by definition of total variation for a vector measure,

$$
|\mu_w|(E) \leqslant \int_E |w|\,d\mu \quad \forall E\in\mathcal{M}\,.
$$

Let us prove the reverse inequality. Let $E\in\mathcal{M}$. If $|\mu_w|(E)=\infty$ we are done. Then suppose $|\mu_w|(E)<\infty$. Note that, from this assumption, we have that each w_i $(i=1,\dots,m)$ is integrable on E, that is $\int_E|w_i|\,d\mu<\infty$. Without loss of generality, we can assume that each w_i is a real-valued function on E and then we can consider $w=(w_1,\dots,w_m) : E\to\mathbb{R}^m$.

Let $D = \{z_h : h \in \mathbb{N}\}$ be a dense set in the unit sphere $\mathbf{S}^{m-1} := \{y \in \mathbb{R}^m : |y| = 1\}$. For any $\epsilon \in (0,1)$ let us define $\sigma : E \to \mathbb{N}$

$$\sigma(x) := \min\{h \in \mathbb{N} : \langle w(x), z_h\rangle \geqslant (1-\epsilon)|w(x)|\} \quad x \in E\,,$$

and let $E_h := \sigma^{-1}(h)$. Then $(E_h)_h \subset \mathcal{M}$ is a pairwise disjoint sequence and $E = \cup_{h=0}^{\infty} E_h$. Therefore

$$(1-\epsilon)\int_E |w|\,d\mu = \sum_{h=0}^{\infty}\int_{E_h}(1-\epsilon)|w|\,d\mu \leqslant$$
$$\sum_{h=0}^{\infty}\int_{E_h}|\langle w(x), z_h\rangle|\,d\mu \leqslant \sum_{h=0}^{\infty}\int_{E_h}|w(x)|\,d\mu \leqslant |\mu_w|(E)\,\forall\epsilon \in (0,1)\,.$$

Thus, getting $\epsilon \to 0$ in the previous inequality, the proof is accomplished.

□

Definition 1.48 (Integrals)**.** Let $(X, \mathcal{M})$ be a measure space.

(i) Let $\nu : \mathcal{M} \to \mathbb{R}$ be a real measure.
If $u : X \to \overline{\mathbb{R}}$ is a $|\nu|$-measurable function, we say that u is *ν-integrable* if u is $|\nu|$-integrable and we set

$$\int_X u\,d\nu := \int_X u\,d\nu^+ - \int_X u\,d\nu^-\,.$$

If $u = (u_1, \dots, u_k) : X \to \overline{\mathbb{R}}^k$ is a $|\nu|$-measurable vector function,we say that u is *ν-integrable* if each its component u_i $(i = 1, \dots, k)$ is $|\nu|$-integrable and we set

$$\int_X u\,d\nu := \left(\int_X u_1\,d\nu, \dots, \int_X u_k\,d\nu\right)\,.$$

(ii) Let $\nu = (\nu_1, \dots, \nu_m) : \mathcal{M} \to \mathbb{R}^m$ be a vector measure.
If $u : X \to \overline{\mathbb{R}}$ is a $|\nu|$-measurable function, we say that u is *ν-integrable* if u is $|\nu|$-integrable and we set

$$\int_X u\,d\nu := \left(\int_X u\,d\nu_1, \dots, \int_X u\,d\nu_m\right)\,.$$

(iii) Let $E \in \mathcal{M}$, the integral of of a function u on E is defined by

$$\int_E u\,d\nu := \int_X u\,\chi_E\,d\nu\,,$$

provided that the right-hand side makes sense.

Remark 1.49. Notice that an immediate consequence of the above definition is the inequality

$$\left|\int_X u\,d\nu\right| \leqslant \int_X |u|\,d|\nu|$$

which holds for every extended real or vector valued summable function u and for every positive, real or vector measure ν.

Definition 1.50 (Absolute continuity and singularity for signed vector measures)**.** Let $(X, \mathcal{M})$ be a measure space.

(i) Let μ be a positive measure and ν be a vector signed measure on the measure space $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ, and write $\nu << \mu$, if $|\nu| << \mu$, as positive measures, that is, for every $E \in \mathcal{M}$ the following implication holds:

$$\mu(E) = 0 \quad \Rightarrow \quad |\nu|(E) = 0\,.$$

(ii) If μ or ν are signed $\overline{\mathbb{R}}^m$-valued signed measures on measure space $(X, \mathcal{M})$, we say that they are mutually singular, and write $\mu \perp \nu$, if $|\mu|$ and $|\nu|$ are mutually singular, as positive measures, that is, there exists $E \in \mathcal{M}$ such that $|\mu|(E) = |\nu|(X \setminus E) = 0$.

Remark 1.51. Observe that, given a positive measure μ on a measure space $(X, \mathcal{M})$, then vector measure μ_w defined in (1.19) is trivially absolutely continuous with respect to μ by Proposition 1.47.

Lebesgue decomposition theorem for vector signed measures 1.52. *Let ν and μ be respectively a $\overline{\mathbb{R}}^m$-valued σ-finite measure and a σ-finite positive measure on a measure space $(X, \mathcal{M})$. Then there is a decomposition of ν such that*

$$\nu = \nu_{ac} + \nu_s\,, \tag{1.21}$$

where ν_{ac} and ν_s are still $\overline{\mathbb{R}}^m$-valued signed measures on $(X, \mathcal{M})$ with $\nu_{ac} << \mu$ and $\nu_s \perp \mu$. The decomposition is unique. Moreover there exists a unique vector function $w = (w_1, \ldots, w_m) : X \to \overline{\mathbb{R}}^m$ with either $w_{i,+}$ or $w_{i,-}$ ($i = 1, \ldots, m$) integrable functions w.r.t. μ such that

$$\nu_{ac}(E) = \mu_w(E) = \int_E w\, d\mu \quad \forall E \in \mathcal{M}\,. \tag{1.22}$$

Proof. Let $\nu = (\nu_1, \ldots, \nu_m) : \mathcal{M} \to \mathbb{R}^m$. By definition, for each $i = 1, \ldots, m$, $\nu_i : \mathcal{M} \to \overline{\mathbb{R}}$ is a σ-finite signed vector measure. By Theorem 1.36, there are two signed measures $\nu_{i,ac}$, $\nu_{i,s} : \mathcal{M} \to \overline{\mathbb{R}}$ such that

$$\nu_{i,ac} << \mu, \quad \nu_{i,s} \perp \mu, \quad \nu_i = \nu_{i,ac} + \nu_{i,s}\,, \tag{1.23}$$

and there exists a measurable function $w_i : X \to \overline{\mathbb{R}}$ such that either $w_{i,+}$ or $w_{i,-}$ is integrable with respect to μ such that

$$\nu_{i,ac}(E) = \int_E w_i\, d\mu \quad \forall\, E \in \mathcal{M}\,. \tag{1.24}$$

Moreover both decomposition (1.23) and representation (1.24) are unique.

Define the signed vector measures ν_{ac}, $\nu_s : \mathcal{M} \to \mathbb{R}^m$

$$\nu_{ac} := (\nu_{1,ac}, \ldots, \nu_{m,ac}) \text{ and } \nu_s := (\nu_{1,s}, \ldots, \nu_{m,s})$$

and the vector function $w := (w_1, \ldots, w_m) : X \to \overline{\mathbb{R}}^m$. Then it is trivial to see that (1.21) and (1.22) now hold for the signed vector measure ν. Therefore the proof is accomplished. $\square$

Each signed vector measure ν is trivially absolutely continuous with respect to its total variation $|\nu|$. The following useful decomposition for vector measures immediately follows from the Lebesgue decomposition theorem for signed vector measures 1.52, Proposition 1.47 and Remark 1.38.

Corollary 1.53 (Polar decomposition for vector measures)**.** *Let ν be a $\mathbb{R}^m$ -valued measure on the measure space $(X, \mathcal{M})$. Then there exists a unique measurable vector function $w_\nu : X \to \overline{\mathbb{R}}^m$ with $|w_\nu(x)| = 1$ $|\nu|$ a.e. $x \in X$ such that $\nu = |\nu|_{w_\nu}$, that is*

$$\nu(E) = \int_E w_\nu \, d|\nu| \quad \forall E \in \mathcal{M}\,.$$

.

Proof. Let us first notice that, by Theorem 1.42 (ii), both ν and $|\nu|$ are finite and $\nu << |\nu|$. By Theorem 1.52 and Remark 1.38 there exists a vector function $w_\nu : X \to \overline{\mathbb{R}}^m$ such that

$$\nu(E) = \int_E w_\nu \, d\,|\nu| \quad \forall E \in \mathcal{M}\,. \tag{1.25}$$

By (1.25) and Proposition 1.47 , we can infer that

$$|\nu|(E) = \int_E |w_\nu| \, d\,|\nu| \quad \forall E \in \mathcal{M}\,.$$

Since $|\nu|$ is finite, we have that $|w_\nu(x)| = 1$ $|\nu|$-a.e. $x \in X$ and the proof is accomplished.

□

1.5. **Spaces $L^p(X, \mu)$ and their main properties. Riesz representation theorem.**

Completeness and dual space of $L^p(X, \mu)$

In this subsection we will only request that $(X, \mathcal{M}, \mu)$ **is a measure space**.

Let us introducel the space of p-integrable functions with respect to measure μ.

Definition 1.54. Let $p \in [1, \infty]$,

$$L^p(X, \mu) := \left\{ f : X \to \overline{\mathbb{R}} : f \text{ is measurable and } \|f\|_{L^p} < +\infty \right\}$$

where

$$\|f\|_{L^p} = \|f\|_{L^p(X,\mu)} := \begin{cases} \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} & \text{if } 1 \leqslant p < \infty \\ \inf \{ M > 0 : |f(x)| \leqslant M \ \mu - \text{a.e. } x \in X \} & \text{if } p = \infty \end{cases} .$$

The quantity $\|f\|_{L^p}$ is called the L^p norm of f on measure space $(X, \mathcal{M}, \mu)$. When $X = \Omega$ is an open subset of $\mathbb{R}^n$, $\mu = \mathcal{L}^n$, $\mathcal{M} = \Omega \cap \mathcal{M}_n$, where $\mathcal{M}_n$ denotes the class of n-dimensional Lebesgue measurable sets of $\mathbb{R}^n$ and d the Euclidean distance, we will simply denote $L^p(X, \mu)$ as $L^p(\Omega)$.

Remark 1.55. When dealing with measure- theoretic or functional-analytic propenies of functions and L^p spaces, it is often convenient to consider functions that agree a.e. as identical, thinking of the elements of L^p spaces as equivalence classes; in particular, this makes $\|\cdot\|_{L^p}$a norm. We shall follow this path whenever our statements will depend only on the equivalence class without further mention, provided that this is clear from the context.

Let us recall the following fundamental result concerning the completeness of L^p (see [GZ, Theorem 6.24]).

Theorem 1.56 (Fisher-Riesz,1907). *$(L^p(X,\mu), \|\cdot\|_{L^p})$ is a B.s. if $1 \leqslant p \leqslant \infty$. Moreover $L^2(X,\mu)$ turns out to be a Hilbert space with respect to the scalar product*

$$(f,g)_{L^2} := \int_X f\, g\, d\mu \quad f,\, g \in L^2(X,\mu)\,.$$

As a consequence of the proof of Riesz- Fisher's Theorem we have the following useful result.

Theorem 1.57. *Let $(f_h)_h \subset L^p(X,\mu)$ and $f \in L^p(X,\mu)$ with $1 \leqslant p \leqslant \infty$. Suppose that*

$$\text{(MC)} \qquad \lim_{h\to\infty} \|f_h - f\|_{L^p(X,\mu)} = 0\,.$$

Then, there exist a subsequence $(f_{h_k})_k$ and a function $g \in L^p(X,\mu)$ such that

(i) $f_{h_k}(x) \to f(x) \quad \mu - a.e.\ x \in X;$
(ii) $|f_{h_k}(x)| \leqslant g(x) \quad \mu - a.e.\ x \in X, \forall\, k.$

Proof. See [GZ, Theorem 6.25]. □

Remark 1.58. The implication (MC)⇒ $f_h(x) \to f(x)$ μ-a.e. $x \in \Omega$ may not hold.

Historical notes: ([P, Sections 1.1.4,1.5.2,4.4.1]) Fisher and Riesz invented the Hilbert space $L^2([a,b])$ in 1907, by proving its completeness. Both authors observed the significance of Lebesgue's integral as the basic ingredient. Subsequently, in 1909, Riesz extended this definition to exponents $1 < p < \infty$ and described how the interval $[a,b]$ can be replaced by any measurable set of $\mathbb{R}^n$. L^p spaces are sometimes called Lebesgue spaces, named after H. Lebesgue (Dunford & Schwartz 1958, III.3), although according to Bourbaki (1987) they were first introduced by Riesz.

Definition 1.59. Let $1 \leqslant p < \infty$ and denote

$$p' := \begin{cases} \dfrac{p}{p-1} & \text{if } 1 < p < \infty \\ \infty & \text{if } p = 1 \end{cases}$$

p' is called conjugate exponent of p.

Theorem 1.60 (Hölder inequality). *Let p and p' be conjugate exponents, $1 \leqslant p < \infty$ Let $f \in L^p(X,\mu)$ and $g \in L^{p'}(X,\mu)$. Then $f\,g \in L^1(X,\mu)$ and*

$$\|fg\|_{L^1(X,\mu)} \leqslant \|f\|_{L^p(X,\mu)}\, \|g\|_{L^{p'}(X,\mu)}$$

Proof. See [R1, Theorem 3.8] or [GZ, Theorem 6.20]. □

The Hölder inequality establishes a duality between $L^p(X,\mu)$ and the dual space of $L^p(X,\mu)$, denoted $(L^p(X,\mu))'$ according to the notation of functional analysis: if $u \in L^{p'}(X,\mu)$, it is well defined the continuous linear functional $T(u) : L^p(X,\mu) \to \mathbb{R}$, that is $T(u) \in (L^p(X,\mu))'$, by

$$T(u)(f) := \langle T(u), f\rangle_{(L^p(X,\mu))'\times L^p(X,\mu)} := \int_X u\, f\, d\mu \quad \forall f \in L^p(X,\mu)\,.$$

The question naturally arises whether all continuous linear functionals on $L^p(X,\mu)$ have this form, and whether the representation is unique. The answer is affirmative if $1 < p < \infty$. It is also affirmative if $p = 1$, provided that an additional condition on measure μ.

Riesz representation theorem for the dual space of L^p 1.61. *If $1 < p < \infty$, then the mapping $T: L^{p'}(X,\mu) \to (L^p(X,\mu))'$, defined by*

$$\langle T(u), f\rangle_{(L^p(X,\mu))'\times L^p(X,\mu)} := \int_X u\, f\, d\mu \quad \forall f \in L^p(X,\mu)\,,$$

is an isometric isomorphism, that is, T is a linear, one-to-one, onto mapping and

$$\|T(u)\|_{(L^p(X,\mu))'} = \|u\|_{L^{p'}(X,\mu)} \quad \forall u \in L^{p'}(X,\mu)\,.$$

If $p = 1$, the same conclusion holds under the additional assumption that μ is σ-finite. We will mean this feature by means of the identification

$$L^{p'}(X,\mu) \equiv (L^p(X,\mu))'\,. \tag{1.26}$$

Proof. See [GZ, Theorem 6.43] or [F, Theorem 6.15], and [R1, Theorem 6.16]) if $p = 1$ provided μ is σ-finite. □

Remark 1.62. Identification (1.26) may fail in the other cases: see [F, section 6.2].

Historical notes:([P, Section 2.2.7]) The identity $(L^p(a,b))' = L^{p'}(a,b)$ with $1 < p < \infty$ was proved by F. Riesz in 1909. The limit case $p = 1$ is due to Steinhuas in 1919.

Density of continuous functions in $(L^p(X,\mu), \|\cdot\|_{L^p})$. Riesz representation theorem.

The subject of this subsection concerns measure and integration theory on locally compact metric spaces. We have seen that the Lebesgue measure on $\mathbb{R}^n$ interacts nicely with the topology on $\mathbb{R}^n$ - measurable sets can be approximated by open or compact sets, and integrable functions can be approximated by continuous functions - and it is of interest to study measures having similar properties on more general spaces. Moreover, it turns out that certain linear functionals on spaces of continuous functions are given by integration against such measures. This fact constitutes an important link between measure theory and functional analysis, and it also provides a powerful tool for constructing measures.

In this subsection we will only request that $(X, \mathcal{B}(X), \mu)$ is a **measure space** with (X, d) **locally compact, separable metric space** (we will often use the abbreviation l.c.s. in the following) and μ a **Radon measure**.

Let us begin to deal with the approximation of continuous functions in L^p. Note that, under the above assumptions, $\mathbf{C}^0_c(X) \subset L^p(X,\mu)$ for each $p \in [1,\infty]$, provided that μ is a Radon measure on X.

Theorem 1.63 (Approximation in L^p by continuous functions)**.** *Let $(X, \mathcal{B}(X), \mu)$ be a measure space with (X, d) l.c.s. and μ Radon measure. Then $C^0_c(X)$ is dense in $(L^p(X), \|\cdot\|_{L^p})$, provided that $1 \leqslant p < \infty$.*

Proof of Theorem 1.63. The proof can be carried out as in the case of $L^p(\Omega)$, by means of the approximation by simple functions (Theorem 1.20) and Lusin's theorem (Theorem 1.23): see [R1, Theorem 3.14] and also [SC, Theorem 2.59] . □

Remark 1.64. Assume that $f \in L^\infty(X,\mu) \cap L^p(X,\mu)$ $(1 \leqslant p < \infty)$. Then by Theorems 1.63 and 1.57, it follows that there exists a sequence $(f_h)_h \subset \mathbf{C}^0_c(X)$ such that

$$f_h \to f \text{ in } L^p(X,\mu) \text{ as } h\to\infty \text{ and } |f_h(x)| \leqslant \|f\|_{L^\infty(X,\mu)} \quad \forall\, x\in X\,.$$

Indeed, by Theorems 1.63 and 1.57, there exist a sequence $(g_h)_h \subset \mathbf{C}^0_c(X)$ and a function $g \in L^p(X,\mu)$ such that

$$g_h \to f \text{ in } L^p(X,\mu) \text{ and } |g_h(x)| \leqslant g(x)\ \mu\text{-a.e. } x\in X\,.$$

. Then, let define $f_h : X \to \mathbb{R}$ as

$$f_h(x) = \min\{\max\{g_h(x), -\|f\|_{L^\infty(X,\mu)}\},\ \|f\|_{L^\infty(X,\mu)}\}$$

We are now going to establish an important relationship between Radon measures and suitable linear bounded (or continuous) functionals on the space of compactly supported continuous functions, called Riesz representation theorem and due to F. Riesz. This relationship will turn out to be a fundamental bridge between measure theory and functional analysis.

Let us first recalls some preliminary topological results.

Urysohn's lemma 1.65 (1925)**.** *Let X be a locally compact metric space, let $K\subset X$ and $V\subset X$ be, respectively, a compact set and an open set such that $K\subset V$. Then there exists a function $\varphi\in\mathbf{C}^0_c(X)$ such that*

$$0\leqslant \varphi\leqslant 1,\quad \varphi\equiv 1 \textit{ in } K \textit{ and } \operatorname{spt}(\varphi)\subset V\,.$$

Proof. We skip the proof (see [GZ, Lemma 9.7]). □

Lemma 1.66 (Partition of unity)**.** *Let (X,d) be a locally compact metric space. Assume that $V_1,\dots,V_N$ and K are respectively open sets and a compact set in X such that*

$$K\subset \cup_{i=1}^N V_i\,.$$

Then there exist $f_i\in\mathbf{C}^0_c(X)$ $(i=1,\dots,N)$ such that

$$\operatorname{spt}(f_i)\subset V_i,\quad 0\leqslant f_i\leqslant 1\quad \forall\, i=1,\dots,N\,;$$

$$\sum_{i=1}^N f_i(x) = 1\quad \forall\, x\in K\,.$$

A family of functions $\{f_1,\dots,f_N\}$ satisfying previous properties is called a **partition of unity subordinate to the open covering** $V_1,\dots,V_N$ **of** K.

Proof. See [GZ, Theorem 9.8] or [R1, Theorem 2.13]. □

Let μ be a Radon measure on measure space $(X,\mathcal{B}(X))$. We can associate to μ a linear functional $L_\mu : \mathbf{C}^0_c(X)\to\mathbb{R}$ defined by

$$L_\mu(u) := \int_X u\,d\mu \quad \forall u\in\mathbf{C}^0_c(X)\,. \tag{1.27}$$

Notice that L_μ is *positive* . Let us recall that

Definition 1.67. A linear functional $L:\mathbf{C}^0_c(X)\to\mathbb{R}$ is said to be *positive* (or also *monotone*) if $L(u)\geqslant 0$ whenever $u\geqslant 0$ (or, equivalently, $L(u)\geqslant L(v)$ whenever $u\geqslant v$).

In this definition there is no mention of continuity, but it is worth noting that positivity itself implies a rather strong continuity property. More precisely, let us endow $(\mathbf{C}_c^0(X))^m$ by the ∞- (or also uniform) norm defined by

$$\|u\|_\infty := \sup_{x\in X} |u(x)|_{\mathbb{R}^m} \quad \text{if } u \in \mathbf{C}_c^0(X)\,. \tag{1.28}$$

For the sake of simplicity, we will denote $\mathbb{R}^m$-norm simply by $|\cdot|$ from now on.

Proposition 1.68. *If $L : \mathbf{C}_c^0(X) \to \mathbb{R}$ is a positive linear functional, for each compact $K \subset X$ there is a positive constant C_K (depending on K) such that*

$$\sup\left\{|L(u)| : u \in \mathbf{C}_c^0(X),\ \|u\|_\infty \leqslant 1, \mathrm{spt}(u) \subset K\right\} \leqslant C_K\,. \tag{1.29}$$

Proof. See [F, Proposition 7.1]. □

We will assume inequality (1.29) as definition of continuity (or boundedness) for linear functionals on the compactly supported continuous functions, even in the vector case $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$.

Definition 1.69. A linear functional $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ is said to be *continuous* (or *bounded*) if, for each compact set $K \subset X$, there is a positive constant C_K (depending on K) such that (1.29) holds.

Remark 1.70. For the sake of simplicity, assume that $m = 1$. Then it can be proved that the notion of continuity in Definition 1.69 is induced by a topology τ on $\mathbf{C}_c^0(X)$ for which $(\mathbf{C}_c^0(X), \tau)$ turns out to be a locally convex , complete topological vector space, which is not metrizable (see [T]). Indeed let $(A_h)_h$ be an increasing sequence of relative compact open sets of (X, d) . Then, if we denote by $\mathbf{C}_0^0(A_h)$ the space of continuous function *vanishing at infinity* on A_h (see Definition 1.80), it can be shown that $(\mathbf{C}_0^0(A_h), \|\cdot\|_\infty)$ is a Banach space (see Proposition 1.81) and $\mathbf{C}_c^0(X) = \cup_{h=1}^\infty \mathbf{C}_0^0(A_h)$. If $i_h : \mathbf{C}_0^0(A_h) \to \mathbf{C}_c^0(X)$ $(h = 1, 2, \dots)$ denotes the inclusion map, the topology τ turns out to be the strongest topology on $\mathbf{C}_c^0(X)$ for which maps $i_h : (\mathbf{C}_0^0(A_h), \|\cdot\|_\infty) \to (\mathbf{C}_c^0(X), \tau)$ are continuous for each h. The existence of such a topology τ can be provided meaning $\mathbf{C}_c^0(X)$ as the vector topological space, also called LF-space, *countable inductive strict limit* of Banach spaces $(\mathbf{C}_0^0(A_h), \|\cdot\|_\infty)_h$ (see, for instance, [T, Chap. XIII]).

Moreover a notion of convergence can be induced by means of this topology.

Exercise: A linear functional $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ is continuous (according to Definition 1.69) if and only if $\lim_{h\to\infty} L(u_h) = L(u)$ for each $(u_h)_h$, u in $(\mathbf{C}_c^0(X))^m$ satisfying

$$u_h \to u \text{ uniformly on } X \text{ and there exists a compact } K \subset X,\ \mathrm{spt}u \cup \bigcup_{h=1}^\infty \mathrm{spt}u_h \subset K\,. \tag{1.30}$$

Definition 1.71. We write that $u_h \to u$ in $(\mathbf{C}_c^0(X))^m$, if (1.30) holds.

Example 1.72. If μ is a Radon measure on X and $w = (w_1, \dots, w_m) : X \to \mathbf{S^{m-1}} := \{y \in \mathbb{R}^m : |y| = 1\}$ is a (Borel) measurable function, we may define a continuous linear functional $w\mu : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ setting

$$w\mu(u) := \int_X (w, u)_{\mathbb{R}^m}\, d\mu = \int_X \sum_{i=1}^m w_i\, u_i\, d\mu \quad u = (u_1, \dots, u_m) \in (\mathbf{C}_c^0(X))^m\,, \tag{1.31}$$

which is trivially an extension of the functional defined in (1.27) when $m = 1$. We will see that each continuous linear functional $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ can be represented by form (1.31) for suitable w and μ.

Riesz representation theorem 1.73. *Let (X, d) be a separable, locally compact metric space and let $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ be a continuous linear functional. Then there exist a Radon measure $\mu_L : \mathcal{B}(X) \to [0, \infty]$ and a Borel measurable vector function $w_L : X \to \mathbf{S}^{m-1}$ such that*

$$(1.32) \qquad L(u) = \int_X (w_L, u)_{\mathbb{R}^m} \, d\mu_L \quad \forall u \in (\mathbf{C}_c^0(X))^m \,,$$

that is, $L = w_L \mu$, and μ_L is characterized by the following identity: for each open set $A \subset X$

$$(1.33) \qquad \mu_L(A) = \sup \{ L(u) : \, u \in (\mathbf{C}_c(X))^m, \, \mathrm{spt} u \subset A, \, \|u\|_\infty \leqslant 1 \} \,.$$

Moreover representation (1.32) *is unique.*

Definition 1.74. Let (X, d) be a locally compact metric space and consider the measure space $(X, \mathcal{B}(X))$. Let us also denote by $\mathcal{B}_{\mathrm{comp}}(X)$ the class of Borel sets which are relatively compact in X.

(i) A $\mathbb{R}^m$-valued Radon vector measure ν on X is a set function $\nu : \mathcal{B}_{\mathrm{comp}}(X) \to \mathbb{R}^m$ for which there exist a Borel vector function $w_\nu : X \to \mathbf{S^{m-1}}$ and a positive Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ such that

$$(1.34) \qquad \nu(E) = \int_E w_\nu \, d\mu \quad \forall E \in \mathcal{B}_{\mathrm{comp}}(X) \,.$$

(ii) A $\mathbb{R}^m$-valued *finite Radon vector measure* is a set function $\nu : \mathcal{B}(X) \to \mathbb{R}^m$ for which there exist a Borel vector function $w_\nu : X \to \mathbf{S^{m-1}}$ and a positive finite Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ such that (1.34) holds for all $E \in \mathcal{B}(X)$.

We will write that $\nu = w_\nu \, \mu$ in both previous cases.

We denote by $(\mathcal{M}_{\mathrm{loc}}(X))^m$ (respectively $(\mathcal{M}(X))^m$) the space of $\mathbb{R}^m$-valued Radon (resp. $\mathbb{R}^m$-valued finite Radon) measures on X.

Remark 1.75. Observe that, by Corollary 1.53, we could equivalently define a $\mathbb{R}^m$-valued Radon vector measure (respectively a $\mathbb{R}^m$-valued finite Radon vector measure) as a set function $\nu : \mathcal{B}_{\mathrm{comp}}(X) \to \mathbb{R}^m$ such that, for each compact $K \subset X$, $\nu : \mathcal{B}(K) \to \mathbb{R}^m$ is a vector measure (respectively as a vector measure $\nu : \mathcal{B}(X) \to \mathbb{R}^m$)

Remark 1.76. It easy to see that $(\mathcal{M}_{\mathrm{loc}}(X))^m$ turns out to be vector space.

Before the proof of the Riesz representation theorem we need some technical lemma.

Lemma 1.77. *Let $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ be a continuous linear functional. Let $\mu_L^* : \mathcal{P}(X) \to [0, \infty]$ be the set function defined as*

$$(1.35) \qquad \mu_L^*(E) := \inf \{ \mu_L(A) : \, A \text{ open}, A \supseteq E \} \qquad \text{if } E \subset X$$

where $\mu_L(A)$ is the quantity in (1.33). *Then μ_L^* is a Radon outer measure on X.*

Proof. Denote $\nu = \mu_L^*$ and $\mu = \mu_L$. Let us first observe that, if U is open, then

$$(1.36) \qquad \mu(U) = \nu(U) \,,$$

that is the definition of ν is consistent. Indeed we immediately get that

$$\mu(U) \geqslant \nu(U)\,.$$

If A is open and $A \supseteq U$, then by definition (see (1.33)), we get

$$\mu(U) \leqslant \mu(A)\,,$$

which implies

$$\mu(U) \leqslant \nu(U)\,.$$

Thus (1.36) follows. Let us now divide the proof in three steps.

1st step. We prove that ν is an outer measure.

Let us first show that ν is countably subaddtive on open sets, that is, if $(A_h)_h$ is a sequence of open sets and $A = \cup_{h=1}^{\infty} A_h$, then

$$\nu(A) \leqslant \sum_{h=1}^{\infty} \nu(A_h)\,. \tag{1.37}$$

Let $u \in (\mathbf{C}_c^0(A))^m$ with $\|u\|_\infty \leqslant 1$ and let $K := \operatorname{spt}(\varphi)$. Since $K \subset A$ is a compact set, there exists an integer N such that $K \subset \cup_{h=1}^{N} A_h$. Let us consider a partition of unity $\{\varphi_1, \dots, \varphi_N\}$ subordinate to $\{A_1, \dots, A_N\}$ and K, that is

$$\varphi_h \in \mathbf{C}_c^0(A_h),\ 0 \leqslant \varphi_h \leqslant 1,\ \sum_{h=1}^{N} \varphi_h(x) = 1\ \forall x \in K\,.$$

Since $u = \sum_{h=1}^{N} u\,\varphi_h$ and $u\varphi_h \in (\mathbf{C}_c^0(A_h))^m$ with $|u\varphi_h| \leqslant 1$

$$L(u) = \sum_{h=1}^{N} L(u\varphi_h) \leqslant \sum_{h=1}^{N} \nu(A_h) \leqslant \sum_{h=1}^{\infty} \nu(A_h)\,.$$

Then (1.37) follows passing to the supremum over all $u \in (\mathbf{C}_c^0(A))^m$ with $\|u\|_\infty \leqslant 1$. Let us now prove that ν is countably subadditive, that is

$$\nu(E) \leqslant \sum_{h=1}^{\infty} \nu(E_h) \quad \text{if } E \subset \cup_{h=1}^{\infty} E_h\,. \tag{1.38}$$

Let us first observe that ν is non decreasing w.r. t. the inclusion, that is

$$\nu(E) \leqslant \nu(F) \text{ if } E \subset F\,.$$

Without loss of generality we can suppose $\nu(E_h) < \infty$ for each h. For each $\epsilon > 0$ and h there is an open set $A_h \supseteq E_h$ such that

$$\nu(A_h) < \nu(E_h) + \frac{\epsilon}{2^h}\,.$$

Thus, by (1.37),

$$\nu(E) \leqslant \nu(\cup_{h=1}^{\infty} A_h) \leqslant \sum_{h=1}^{\infty} \nu(A_h) \leqslant \sum_{h=1}^{\infty} \nu(E_h) + \epsilon \quad \forall \epsilon > 0\,.$$

Therefore (1.38) follows.

2nd step. Let us prove that ν is a Borel regular outer measure. In order to prove that ν is a Borel outer measure, by Carathéodory's criterion (see Theorem 1.5 (ii)), we have only to prove that

$$\nu(E \cup F) \geqslant \nu(E) + \nu(F) \quad \text{whenever } d(E,F) > 0\,. \tag{1.39}$$

Assume E, F are open, and let $\varphi \in (\mathbf{C}^0_c(E \cup F))^m$ with $|\varphi| \leqslant 1$. Then, since $\bar{E}$ and $\bar{F}$ are disjoint,

$$\varphi = \varphi|_E + \varphi|_F,\ \varphi|_E \in (\mathbf{C}^0_c(E))^m,\ \varphi|_F \in (\mathbf{C}^0_c(F))^m,\ |\varphi|_E| \leqslant 1,\ |\varphi|_F| \leqslant 1\,.$$

Thus

$$L(\varphi) = L(\varphi|_E) + L(\varphi|_F) \geqslant \nu(E) + \nu(F)\,,$$

and (1.39) follows. In the general case, since $0 < d(E,F) = d(\bar{E},\bar{F})$, there exist open sets A_1, A_2 such that $E \subset A_1$ and $F \subset A_2$ with $d(A_1,A_2) > 0$. If A is open and $E \cup F \subset A$, then $d(A_1 \cap A, A_2 \cap A) > 0$ and $E \subset A_1 \cap A$, $F \subset A_2 \cap A$, so that (1.39)on open sets implies

$$\nu(A) \geqslant \nu\left((A_1 \cap A) \cup (A_2 \cap A)\right) \geqslant \nu(A_1 \cap A) + \nu(A_2 \cap A) \geqslant \nu(E) + \nu(F)\,.$$

As A is arbitrary, (1.39) follows. Hence ν is a Borel measure. Moreover ν is Borel regular, since, if $E \subset X$, $\nu(E) < \infty$ and $(A_h)_h$ are open sets with $E \subset A_h$ and $\lim_{h\to\infty} \nu(A_h) = \nu(E)$, then $B := \cap_{h=1}^{\infty} A_h$ is a Borel set with $E \subset B$ and $\nu(E) = \nu(B)$. If $\nu(E) = \infty$, we can choose as Borel envelope $B = X$.

3rd step. Let us prove that ν is finite on compact sets. Let us recall, that by Lemma 1.17, there exists an increasing sequence of open sets $(V_i)_i$ such that

$$X = \cup_{i=1}^{\infty} V_i, \quad \overline{V_i} \text{ compact for each } i\,.$$

In particular, since L is bounded, by (1.33) and (1.29), it follows that

$$\nu(V_i) = \mu(V_i) < \infty \quad \forall\, i\,.$$

Given a compact set $K \subset X$, then there exists an integer i_0 such that

$$K \subset \cup_{i=1}^{i_0} V_i\,.$$

Therefore, since ν is subadditive,

$$\nu(K) \leqslant \sum_{i=1}^{i_0} \nu(V_i) < \infty\,,$$

and the proof is accomplished. □

Proof of the Riesz representation theorem 1.73. By Lemma 1.77, $\nu := \mu^*_L$ is a Radon outer measure on X. Let us now define functional $\tilde{L} : \mathbf{C}^0_c(X;[0,\infty)) \to [0,\infty)$ as

$$\tilde{L}(\varphi) := \sup\left\{L(u) :\ u \in\ (\mathbf{C}^0_c(X))^m,\ |u| \leqslant \varphi, \varphi \in \mathbf{C}^0_c(X;[0,\infty))\right\}\,.$$

We will divide the proof in three steps. In step one we will show that $\tilde{L}$ is additive, positively homogeneous of degree one, and monotone on $\mathbf{C}^0_c(X;[0,\infty))$. In step two, we will show the inequality

$$\tilde{L}(\varphi) \leqslant \int_X \varphi\, d\nu \quad \forall\, \varphi \in \mathbf{C}^0_c(X;[0,\infty))\,. \tag{1.40}$$

Finally, in step three, by using the Riesz representation theorem for the dual of $L^1(X,\mu)$ (see Theorem 1.61) we can conclude the proof.

1st step. We show that, whenever $\varphi_i \in \mathbf{C}^0_c(X;[0,\infty))$ $(i=1,2)$ and $c \geqslant 0$, we have

$$\tilde{L}(\varphi_1+\varphi_2) = \tilde{L}(\varphi_1) + \tilde{L}(\varphi_2)\,, \tag{1.41}$$

$$\tilde{L}(c\,\varphi_1) = c\,\tilde{L}(\varphi_1), \tag{1.42}$$

$$\tilde{L}(\varphi_1) \leqslant \tilde{L}(\varphi_2), \quad \text{if } \varphi_1 \leqslant \varphi_2\,. \tag{1.43}$$

It si immediate that (1.43) follows by the definition of $\tilde{L}$. let us prove the remaining properties. By definition of $\tilde{L}$ and the linearity of L, for each $u_i \in (\mathbf{C}^0_c(X))^m$ $(i=1,2)$ with $|u_i| \leqslant \varphi_i$, $c \geqslant 0$, we can infer

$$\begin{gathered}\tilde{L}(\varphi_1+\varphi_2) \geqslant L(u_1+u_2) = L(u_1) + L(u_2)\,,\\ \tilde{L}(c\,\varphi_1) \geqslant L(c\,u_1) = c\,L(u_1)\,,\\ L(c\,\psi_1) = c\,L(\psi_1) \leqslant c\,\tilde{L}(\varphi_1)\,,\\ L(u_i) \leqslant \tilde{L}(\varphi_i) \quad i=1,2,\,.\end{gathered}$$

Then, by the previous inequalities, it follows respectively that

$$\tilde{L}(\varphi_1+\varphi_2) \geqslant \tilde{L}(\varphi_1) + \tilde{L}(\varphi_2)\,, \tag{1.44}$$

$$\tilde{L}(c\,\varphi_1) \geqslant c\,\tilde{L}(\varphi_1)\,, \tag{1.45}$$

$$\tilde{L}(c\,\varphi_1) \leqslant c\,\tilde{L}(\varphi_1)\,. \tag{1.46}$$

Therefore, by (1.45) and (1.46) we have proven (1.42), and we have only to prove the inverse inequality of (1.44) for showing (1.41) . Now let $u \in (\mathbf{C}^0_c(X))^m$ be such that $|u| \leqslant \varphi_1+\varphi_2$, and set

$$u_i := \begin{cases} \dfrac{\varphi_i}{\varphi_1+\varphi_2}\,u & \text{on } \{\varphi_1+\varphi_2 > 0\} \\ 0 & \text{otherwise}\end{cases} \qquad i=1,2\,.$$

Exercise: Prove that $u_i \in (\mathbf{C}^0_c(X))^m$, $|u_i| \leqslant \varphi_i$ $(i=1,2)$ and $u = u_1+u_2$.

Therefore

$$L(u) = L(u_1) + L(u_2) \leqslant \tilde{L}(\varphi_1) + \tilde{L}(\varphi_2),$$

and complete the proof by the arbitrariness of u.

2nd step. Let us now prove (1.40). Given $\varphi \in \mathbf{C}^0_c(X;[0,\infty))$ and $\epsilon > 0$, let $t_0,\dots,t_N$ be real numbers such that

$$t_0 < 0 < t_1 < \cdots < t_{N-1} < \sup_X \varphi < t_N, \quad t_{h+1}-t_h \leqslant \epsilon \text{ with } h=1,\dots,N-1 \tag{1.47}$$

and consider the partition $\{E_1,\dots,E_N\}$ of $\mathrm{spt}(\varphi)$ by disjoint Borel sets, defined as

$$E_h = \{x \in \mathrm{spt}(\varphi) : \, t_{h-1} < \varphi(x) \leqslant t_h\}\,, \quad 1 \leqslant h \leqslant N\,.$$

Since ν is a Radon o.m., by Theorem 1.14, there exist open sets A_h with $E_h \subset A_h$ and

$$\nu(A_h) \leqslant \nu(E_h) + \frac{\epsilon}{N}, \quad 1 \leqslant h \leqslant N\,. \tag{1.48}$$

If necessary replacing A_h with $\{x \in A_h : \varphi(x) < t_h + \epsilon\}$, we can also assume

$$\varphi < t_h + \epsilon \text{ on } A_h \,. \tag{1.49}$$

Finally, let $\{f_1, \dots, f_N\}$ be a partition of unity subordinated to the open covering $A_1, \dots, A_N$ of the compact set $\mathrm{spt}(\varphi)$, namely $f_h \in \mathbf{C}^0_c(A_h)$,$0 \leqslant f_h \leqslant 1$, and $\sum_{h=1}^N f_h(x) = 1$ on $\mathrm{spt}(\varphi)$. Since $\varphi = \sum_{h=1}^N f_h \, \varphi$, by step 1 and (1.49), we find that

$$\tilde{L}(\varphi) = \sum_{h=1}^N \tilde{L}(f_h \varphi) \leqslant \sum_{h=1}^N (t_h + \epsilon)\, \tilde{L}(f_h)\,. \tag{1.50}$$

If $\psi \in (\mathbf{C}^0_c(X))^m$, and $|\psi| \leqslant f_h$, then $\mathrm{spt}(\psi) \subset A_h$ and $|\psi| \leqslant 1$. Hence, $\tilde{L}(f_h) \leqslant \nu(A_h)$ and, by (1.47) and (1.48), we find that

$$\begin{aligned}
\tilde{L}(\varphi) &\leqslant \sum_{h=1}^N (t_h + \epsilon) \left(\nu(E_h) + \frac{\epsilon}{N} \right) \\
&\leqslant \sum_{h=1}^N (t_{h-1} + 2\epsilon) \left(\nu(E_h) + \frac{\epsilon}{N} \right) \\
&= \sum_{h=1}^N t_{h-1}\, \nu(E_h) + \frac{\epsilon}{N} \sum_{h=1}^N t_{h-1} + 2\epsilon \sum_{h=1}^N \nu(E_h) + 2\epsilon^2 \\
&\leqslant \int_X \varphi \, d\nu + t_N \epsilon + 2\epsilon\, \nu(\mathrm{spt}(\varphi)) + 2\epsilon^2 \\
&\leqslant \int_X \varphi \, d\nu + \epsilon \left(\sup_X \varphi + \epsilon + 2\, \nu(\mathrm{spt}(\varphi)) + 2\epsilon \right)
\end{aligned}$$

If we let $\epsilon \to 0^+$ in the previous inequality, (1.40) follows.

3rd step. Given $e \in \mathbf{S}^{m-1}$, we define $L_e : \mathbf{C}^0_c(X) \to \mathbb{R}$ by

$$L_e(\varphi) := L(\varphi\, e) \quad \varphi \in \mathbf{C}^0_c(X)\,.$$

By (1.40), we find that, for every $\varphi \in \mathbf{C}^0_c(X)$,

$$\begin{aligned}
L_e(\varphi) &\leqslant \sup \left\{ L(\psi) : \psi \in (\mathbf{C}^0_c(X))^m, \, |\psi| \leqslant |\varphi| \right\} \\
&= \tilde{L}(|\varphi|) \leqslant \int_X |\varphi| \, d\nu \,.
\end{aligned}$$

By the approximation in L^p by continuous functions (see Theorem 1.63), we may extend L_e as a linear functional $L_e : L^1(X, \nu) \to \mathbb{R}$ such that

$$|L_e(u)| \leqslant \int_X |u| \, d\nu = \|u\|_{L^1(X,\mu)} \quad \forall\, u \in L^1(X, \nu)\,.$$

Thus, by the Riesz representation theorem for the dual of $L^1(X, \nu)$, there exists $w_e \in L^\infty(X, \nu)$ such that

$$L(u\, e) = \int_X u\, w_e \, d\nu \quad \forall\, u \in L^1(X, \nu)\,.$$

If we set $w_L := (w_1, \ldots, w_m) : X \to \mathbb{R}^m$, $w_i := w_{e_i}$, where $\{e_1, \ldots, e_m\}$denotes the standard basis of $\mathbb{R}^m$, then w_L is bounded and ν-measurable, with

$$L(\varphi) = \sum_{i=1}^{m} L_{e_i}(\varphi_i) = \sum_{i=1}^{m} \int_X w_i\, \varphi_i\, d\nu = \int_X (w_L, \varphi)_{\mathbb{R}^m}\, d\nu \tag{1.51}$$

for every $\varphi = (\varphi_1, \ldots, \varphi_m) \in (\mathbf{C}_c^0(X))^m$. Let us now prove that

$$|w_L(x)| = 1 \quad \nu\text{-a.e. } x \in X\,. \tag{1.52}$$

By (1.36) and (1.51), it follows that, for each bounded open set $A \subset X$,

$$\begin{aligned} \nu(A) &= \sup\{L(\varphi) : \varphi \in (\mathbf{C}_c(X))^m,\ \mathrm{spt}\varphi \subset A,\ |\varphi| \leqslant 1\} \\ &= \sup\left\{\int_X (w_L, \varphi)_{\mathbb{R}^m}\, d\nu : \varphi \in (\mathbf{C}_c(X))^m,\ \mathrm{spt}\varphi \subset A,\ |\varphi| \leqslant 1\right\} \\ &\leqslant \int_A |w_L|\, d\nu\,. \end{aligned} \tag{1.53}$$

From (1.53), it follows that

$$|w_L| > 0\text{-a.e. in } X \text{ and } u := \chi_{\{|w_L|>0\}} \frac{w_L}{|w_L|} \in (L^1(A,\nu))^m\,.$$

By Remark 1.64, there exists a sequence $(\varphi_h)_h \subset (\mathbf{C}_c^0(A))^m$ such that

$$\varphi_h \to u \text{ in } (L^1(A,\nu))^m \text{ and } |\varphi_h| \leqslant 1\,.$$

This implies that

$$(w_L, \varphi_h)_{\mathbb{R}^m} \to |w_L| \text{ in } L^1(A,\nu) \tag{1.54}$$

Therefore, since

$$\nu(A) \geqslant \int_A (w_L, \varphi_h)_{\mathbb{R}^m}\, d\nu\,,$$

by (1.54) and passing to the limit, as $h \to \infty$, in the previous inequality, it follows that

$$\nu(A) \geqslant \int_A |w_L|\, d\mu\,. \tag{1.55}$$

Thus, by (1.53) and (1.55), we can infer that

$$\nu(A) = \int_A |w_L|\, d\nu \text{ for each bounded open set } A \subset X$$

which implies (1.52).

Let us prove that we can choose $w_L : X \to \mathbf{S}^{m-1}$ as a Borel measurable vector function. Indeed, observe that, by Corollary 1.24, we can assume that $w_L : X \to \mathbb{R}^m$ is Borel measurable and satisfying (1.51). On the other hand, by (1.52), there exists a ν-null set $N \subset X$ (which, a priori, could be not a Borel set) such that $|w_L(x)| = 1$ for each $x \in X \setminus N$. Since ν is Borel regular, there exists a Borel ν-null set $B \supset N$. Thus, by changing the value of w_L on B, for instance putting $w_L := e_1$ on B, we get that $w_L : X \to \mathbf{S}^{m-1}$ is still Borel measurable and satisfies (1.51). The uniqueness of ν and w_L (ν-a-e.) follows in a standard way.

Finally, without loss of generality, since $\nu(A) = \mu_L^*(A) = \mu_L(A)$ for each open set $A \subset X$, we still denote μ_L the outer measure defined in (1.35) and yield a Radon measure $\mu_L : \mathcal{B}(X) \to [0,\infty]$ satisfying the desired properties.

□

Riesz representation theorem 1.73 provides a characterization of the measures spaces $\mathcal{M}(X)^m$ and $(\mathcal{M}_{\rm loc}(X))^m$ as dual spaces of suitable spaces of continuous functions.

Observe first that functional (1.27) can make sense even for Radon signed measures. Indeed, if $\nu \in \mathcal{M}_{\rm loc}(X)$, that is, $\nu = w_\nu\,\mu$ with μ positive Radon measure on X and $w_\nu : X \to \{-1,1\}$ Borel function, then , according also to Definition 1.48 (i), it is well defined

$$L_\nu(u) := \int_X u\,d\nu = \int_X w_\nu\,u\,d\mu \quad \forall u \in \mathbf{C}^0_c(X)\,. \tag{1.56}$$

The functional $L_\nu : \mathbf{C}^0_c(X) \to \mathbb{R}$ still turns out to be a linear continuous functional according to Definition 1.69. Thus a trivial consequence of Theorem 1.73 is the following

Corollary 1.78 (Characterization of $\mathcal{M}_{\rm loc}(X)$)**.** *Let (X,d) be a locally compact separable metric space and define*

$$(C^0_c(X))' := \left\{L : C^0_c(X) \to \mathbb{R} : L \textit{ is linear and continuous w.r.t. Definition 1.69} \right\}. \tag{1.57}$$

Let us define the map

$$I : \mathcal{M}_{\rm loc}(X) \to (C^0_c(X))' \quad I(\nu) := L_\nu\,. \tag{1.58}$$

Then I is an isomorphism (between vector spaces).

If $\nu \in \mathcal{M}(X)$ functional (1.56) is still well defined since $\mathcal{M}(X) \subset \mathcal{M}_{\rm loc}(X)$. In this case the functional is actually an element of the dual of normed vector space $(\mathbf{C}^0_c(X), \|\cdot\|_\infty)$ and we will write $L_\nu \in (\mathbf{C}^0_c(X), \|\cdot\|_\infty)'$.

Proposition 1.79. *Let $\nu \in \mathcal{M}(X)$, and let $L_\nu : \mathbf{C}^0_c(X) \to \mathbb{R}$ be the functional in* (1.56)*. Then*

$$\|L_\nu\|_{(\mathbf{C}^0_c(X),\|\cdot\|_\infty)'} := \sup\left\{|L_\nu(u)| : u \in \mathbf{C}^0_c(X),\ \|u\|_\infty \leqslant 1\right\} = |\nu|(X)\,.$$

Proof. It is a special case of (1.33). □

Functional L_ν can actually be extended to the completion of $(\mathbf{C}^0_c(X), \|\cdot\|_\infty)$, that is the class of continuous functions on (X,d) *vanishing at infinity.*

Definition 1.80 (Functions vanishing at infinity)**.** Let (X,d) be a locally compact metric space.

(i) A function $u : X \to \mathbb{R}$ is said to *vanish at infinity* if for every $\epsilon > 0$ there is a compact set $K \subset X$ such that

$$|u(x)| < \epsilon \quad \forall x \in X \setminus K\,.$$

(ii) The class of all continuous $u : X \to \mathbb{R}$ which vanish at infinity is called $C^0_0(X)$, It is clear that $C_c^0(X) \subset C_0^0(X)$ and that the two classes coincide if (X, d) is compact. More precisely, if (X, d) is compact, then

$$C_c^0(X) = C_0^0(X) = C^0(X).$$

It is also well known that

Proposition 1.81. *$C_0^0(X)$ is the completion of normed vector space $(\mathbf{C}_c^0(X), \|\cdot\|_\infty)$.*

Proof. See [R1, Theorem 3.17]. □

Proposition 1.82. *Let $\nu \in \mathcal{M}(X)$ and define*

$$\|\nu\| := |\nu|(X). \tag{1.59}$$

Then $(\mathcal{M}(X), \|\cdot\|)$ is a real normed vector spaces.

Proof. [F, Proposition 7.16]. □

Theorem 1.83 (Characterization of $\mathcal{M}(X)$)**.** *Let (X, d) be a locally compact separable metric space. Let I the map in* (1.58)*. Then*

(i) *$I(\mathcal{M}(X)) = (C_0^0(X), \|\cdot\|_\infty)'$;*

(iI) *$I : (\mathcal{M}(X), \|\cdot\|) \to (C_0^0(X), \|\cdot\|_\infty)'$ is a topological isomorphism, that is an algebraic isomorphism, continuous with its inverse.*

Proof. [R1, Theorem 6.19]. □

Corollary 1.84. *Let (X, d) be a compact metric space. Then $(C^0(X), \|\cdot\|_\infty)'$ is isometrically isomporphic to $\mathcal{M}(X)$.*

<u>Compactness in $(L^p(\Omega), \|\cdot\|_{L^p})$.</u>

In this section we are going to deal with some compactness results in L^p spaces. We will only state these results, without proofs which can be found in [B, Section 4.5].

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $v \in \mathbb{R}^n$ be given, then we define by $\tau_v f : \mathbb{R}^n \to \overline{\mathbb{R}}$ the v-translated function of f defined by

$$(\tau_v f)(x) := f(x + v).$$

Theorem 1.85 (M. Riesz- Fréchét- Kolmogorov)**.** *Let $\mathcal{F}$ be a <u>bounded</u> subset in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ with $1 \leqslant p < \infty$. Suppose that $\lim_{v\to 0} \|\tau_v f - f\|_{L^p} = 0$ uniformly for $f \in \mathcal{F}$, that is*

$$\forall \epsilon > 0\, \exists \delta = \delta(\epsilon) > 0 \text{ such that } \|\tau_v f - f\|_{L^p} < \epsilon \quad \forall v \in \mathbb{R}^n \text{ with } |v| < \delta, \quad \forall f \in \mathcal{F}. \tag{$EN_{\mathcal{F}}$}$$

Then $\mathcal{F}|_\Omega := \{f|_\Omega : f \in \mathcal{F}\}$ is relatively compact in $(L^p(\Omega), \|\cdot\|_{L^p})$, i.e. its closure is compact in $(L^p(\Omega), \|\cdot\|_{L^p})$, for each open set $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure.

From Theorem 1.85 it follows the following compactness criterion in $(L^p(\Omega), \|\cdot\|_{L^p})$. If $f : \Omega \to \overline{\mathbb{R}}$, let us denote by $\tilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ the function defined as

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

Corollary 1.86. *Let $\Omega \subset \mathbb{R}^n$ be an open set with finite measure, let $\mathcal{F} \subset L^p(\Omega)$ and let $\tilde{\mathcal{F}} := \{\tilde{f} : f \in \mathcal{F}\}$. Assume that*

(i) *$\mathcal{F}$ is bounded in $(L^p(\Omega), \|\cdot\|_{L^p})$ with $1 \leqslant p < \infty$;*

(ii) $\lim_{v\to 0} \|\tau_v \tilde{f} - \tilde{f}\|_{L^p} = 0$ *uniformly for $f \in \mathcal{F}$, that is, $\tilde{\mathcal{F}}$ satisfies $EN_{\tilde{\mathcal{F}}}$.*

Then $\mathcal{F}$ is relatively compact in $(L^p(\Omega), \|\cdot\|_{L^p})$.

Proof. From Theorem 1.85, $\tilde{\mathcal{F}}$ is relatively compact. Notice now that $\tilde{\mathcal{F}}$ is relatively sequentially compact in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ if and only if $\mathcal{F}$ is relatively sequentially compact in $(L^p(\Omega), \|\cdot\|_{L^p})$. Thus, the characterization of compact sets in metric spaces completes the proof. □

Eventually recall the following characterization of compactness in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$.

Theorem 1.87. *Let $\mathcal{F} \subset L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$. Then $\mathcal{F}$ is relatively compact in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ if and only if*

(i) *$\mathcal{F}$ is bounded in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$;*

(ii) *for each $\epsilon > 0$ there exists $r_\epsilon > 0$ such that*

$$\|f\|_{L^p(\mathbb{R}^n \setminus B(0,r_\epsilon))} < \epsilon \quad \forall f \in \mathcal{F};$$

(iii) $\lim_{v\to 0} \|\tau_v f - f\|_{L^p} = 0$ *uniformly for $f \in \mathcal{F}$.*

Remark 1.88. (i) The assumption $(EN_{\mathcal{F}})$ is necessary in Theorem 1.85. Indeed, consider the family $\mathcal{F} := \{f_h : h \in \mathbb{N}\}$ where $f_h : \mathbb{R} \to \mathbb{R}$ is defined as $f_h(x) := \begin{cases} h & \text{if } 0 \leqslant x \leqslant \dfrac{1}{h} \\ 0 & \text{otherwise} \end{cases}$ and let $\Omega := (0,1)$. Then it is easy to see that $\|f_h\|_{L^1(\mathbb{R})} = 1$ for each $h \in \mathbb{N}$ and $\mathcal{F}|_\Omega$ is not relatively compact in $(L^1(\Omega), \|\cdot\|_{L^1})$, since there are no subsequences of $(f_h)_h$ converging in $L^1(\Omega)$ (see Exercise III.6). On the other hand, for given $v > 0$, for each $h > \frac{1}{v}$

$$\|\tau_v f_h - f_h\|_{L^1(\mathbb{R})} \geqslant \int_{-\infty}^{0} f_h(x+v)\, dx = \int_0^v f_h(x) = 1\,.$$

Thus, $(EN_{\mathcal{F}})$ does not hold for $\mathcal{F}$.

(ii) If Ω has not finite measure, then the conclusions of Theorem 1.85 need not hold. Indeed , consider the family $\mathcal{F} := \{f_h : h \in \mathbb{N}\}$ where $f_h : \mathbb{R} \to \mathbb{R}$ is defined as $f_h(x) := f(x+h)$ where $f \in Lip(\mathbb{R})$ with $\mathrm{spt}(f) = [-a,a]$, $a > 0$, and f not identically vanishing. Then

$$\|f_h\|_{L^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} > 0 \quad \forall h\,. \tag{1.60}$$

Moreover $\mathcal{F}$ satisfies $(EN_{\mathcal{F}})$, because

$$|\tau_v f(x) - f(x)| = |f(x+v) - f(x)| \leqslant L\,|v|\,\chi_{[-a-1,a+1]}(x) \quad \forall x \in \mathbb{R},\ v \in [-1,1]\,,$$

and

$$\|\tau_v f_h - f_h\|_{L^1(\mathbb{R})} = \|\tau_v f - f\|_{L^1(\mathbb{R})} \quad \forall h$$

where $L := \mathrm{Lip}(f)$. Let $\Omega := \mathbb{R}$ and observe now that $\mathcal{F} = \mathcal{F}|_\Omega$ is not relatively compact in $(L^1(\mathbb{R}), \|\cdot\|_{L^1})$. Otherwise a contradiction arises by (1.60), since $f_h(x) \to 0$ for each $x \in \mathbb{R}$,

Historical notes:[HOH] A first compactness type-result was proved by Fréchet in 1908 in the setting of l^2. In 1931, Kolmogorov proved the first result in this direction. The result characterizes the compactness in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, in the case where all functions are supported in a common bounded set. Condition (iii) of Theorem 1.87 is replaced by the uniform convergence in L^p norm of spherical means of each function in the class to the function itself. (Clearly, our condition (ii) is automatic in this case.) Just a year later, Tamarkin expanded this result to the case of unbounded supports by adding condition (ii) of Theorem 1.87. In 1933, Tulajkov expanded the Kolmogorov-Tamarkin result to the case $p = 1$. In the same year, and probably independently, Riesz proved the result for $1 < p < \infty$, essentially in the form of our Theorem 1.87. Thus we feel somewhat justified in using the names Kolmogorov and Riesz in referring to the theorem, though we are perhaps being a bit unfair to Tamarkin and Tulajkov in doing so. In 1937, M.Fréchét replaced conditions (i) and (ii) of Theorem 1.87 with a single condition ("equisummability"), and generalized the theorem to arbitrary positive p.

1.6. **Operations on measures.** In this section we discuss some useful and fundamental operations on measures and related notions: among them, we describe the product measures, and state the related Fubini and Tonelli theorems, together with some consequences.

Definition 1.89 (Support)**.** Let μ be a positive measure on a separable metric space X; we call the closed set of all points $x \in X$ such that $\mu(U) > 0$ for every neighbourhood U of x the *support* of μ denoted $\mathrm{spt}(\mu)$. In other words,

$$\begin{aligned} \mathrm{spt}\mu = {} & X \setminus \cup\{V : V \text{ open}, \mu(V) = 0\} \\ = {} & X \setminus \{x \in X : \exists\, r > 0 \text{ such that } \mu(B(x,r)) = 0\} \end{aligned} \tag{1.61}$$

If ν is a signed or signed vector measure, we call the support of ν the support of $|\nu|$.

In the general case of a measure on a measure space $(X, \mathcal{M})$, we say that ν is *concentrated* on $S \subset X$ if $S \in \mathcal{M}$ and $|\nu|(X \setminus S) = 0$. Notice that it is impossible in general to define a "minimal" set where a measure is concentrated, hence the set S is not uniquely determined. However, for any pair ν_1 and ν_2 of mutually singular measures there exist pairwise disjoint $\mathcal{M}$-measurable sets S_1 and S_2 such that ν_i is concentrated on S_i $(i = 1, 2)$. A property which is not shared by the support.

Example: Consider, for instance, the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and measures $\mu_1 := \mathcal{L}^1$ and $\mu_2 := \sum_h 2^{-h}\, \delta_{x_h}$ with $(x_h)_h$ dense in $\mathbb{R}$. Then $\mathrm{spt}(\mu_1) = \mathrm{spt}(\mu_2) = \mathbb{R}$. Even if measure μ_2 is concentrated on the elements of the sequence, but $\mathrm{spt}(\mu_1)$ contains every accumulation point of the sequence.

Remark 1.90. Notice also that, if X is a separable metric space and ν is a Borel measure on X, then $\mathrm{spt}(\nu)$ is the smallest closed set where ν is concentrated.

Definition 1.91 (Restriction)**.** Let ν be an outer measure X or a positive or vector signed measure on the measure space $(X, \mathcal{M})$. Let $E \subset X$.

(i) If ν is an outer measure on X, the restricted outer measure of ν to E, written $\nu \llcorner E$, is the set function defined by

$$\nu \llcorner E(F) := \nu(E \cap F) \quad \forall F \subset X . \tag{1.62}$$

(ii) If ν is a positive or vector signed measure on the measure space $(X, \mathcal{M})$ and $E \in \mathcal{M}$, the restricted measure of ν to E, still written $\nu \llcorner E$, is the set function defined in (1.94), but with $F \in \mathcal{M}$.

Theorem 1.92. (i) *If ν is an outer measure on X, then so is $\nu \llcorner E$. Moreover every ν-measurable set is also $\nu \llcorner E$-measurable.*

(ii) *If ν is a Borel regular outer measure on X and E is ν-measurable with $\nu(E) < \infty$, then $\nu \llcorner E$ is a Radon outer measure.*

(iii) *If ν is a positive or vector signed measure on a measure space $(X, \mathcal{M})$ and $E \in \mathcal{M}$, so is $\nu \llcorner E$.*

Proof. (i) It is clear that $\nu \llcorner E$ is an outer measure. By using the definition of Carathéodory measurability, the second part of the statement follows, too. Note that E can be arbitrary here.

(ii) Clearly $(\nu \llcorner E)(K) \leqslant \nu(E) < \infty$ for each compact set $K \subset X$. Since, by previous claim (i), every ν-measurable is also $(\nu \llcorner E)$-measurable, $(\nu \llcorner E)$ is a Borel outer measure. Thus we have only to show that $\nu \llcorner E$ is a Borel regular outer measure.

Since ν is Borel regular, there is a Borel set B such that $E \subset B$ and $\nu(B) = \nu(E)$. Then, since E is ν-measurable and $\nu(E) < \infty$, $\nu(B \setminus E) = 0$. Let us fix $C \subset X$. Then

$$\begin{aligned}(\nu \llcorner E)(C) &\leqslant (\nu \llcorner B)(C) = \nu(C \cap B) = \nu(C \cap B \cap E) + \nu((C \cap B) \setminus E) \\ &\leqslant \nu(C \cap E) + \nu(B \setminus E) = (\nu \llcorner E)(C)\end{aligned}$$

Therefore $\nu \llcorner B = \nu \llcorner E$, so we may as well assume that E is a Borel set.

Given $C \subset X$, we must show that there exists a Borel set F such that $C \subset F$ and $(\nu \llcorner E)(F) = (\nu \llcorner E)(C)$. Since ν is Borel regular, there is a Borel set D such that $C \cap E \subset D$ and $\nu(C \cap E) = \nu(D)$. Let $F := D \cup (X \setminus E)$. Since D and E are Borel sets, so is F. Moreover $C \subset (E \cap C) \cup (X \setminus E) \subset F$. Finally, since $F \cap E = D \cap E$,

$$\begin{aligned}(\nu \llcorner E)(F) &= \nu(F \cap E) = \nu(D \cap E) \leqslant \nu(D) \\ &= \nu(C \cap E) = (\nu \llcorner)E(C) .\end{aligned}$$

Thus $(\nu \llcorner E)(F) = (\nu \llcorner E)(C)$ and so $\nu \llcorner E$ is Borel regular.

(iii) The claim is trivial. □

Remark 1.93. By using the same proof of Theorem 1.92 (ii), we can infer that if ν is a Borel regular measure and E is a Borel set, then $\nu \llcorner E$ is still Borel regular, even if $\nu(E) = \infty$.

Theorem 1.92 (ii) allows to generate Radon outer measures by restricting a given Borel regular outer measure to a measurable set of finite measure. An other interesting procedure for generating Radon outer measures, in a separable, locally compact metric space, by restriction of a given Borel regular outer measure is the following.

Theorem 1.94. *Let ν be a Borel regular outer measure on a separable, locally compact metric space (X, d). Let E be a ν-measurable set such that $\nu \llcorner E$ is a locally finiteouter measure. Then $\nu \llcorner E$ is a Radon outer measure.*

Proof. Let us first recall that, from Lemma 1.17, there exists an increasing sequence of open sets $(V_i)_i$ such that

$$X = \cup_{i=1}^{\infty} V_i, \quad \overline{V_i} \text{ is compact for each } i\,. \tag{1.63}$$

Let $\varphi := \nu \llcorner E$. Let us first prove that

$$\varphi(K) < \infty \quad \text{for each compact set } K \subset X\,. \tag{1.64}$$

If $K \subset X$ is a given compact set, by the local fineteness of φ, for each $x \in K$ there exists an open ball $U(x, r_x)$ such that

$$\varphi(U(x, r_x)) < \infty\,. \tag{1.65}$$

Since K is compact, there is a finite family of open balls $U(x_1, r_1), \ldots, U(x_m, r_m)$ such that

$$K \subset \cup_{i=1}^{m} U(x_i, r_i)\,. \tag{1.66}$$

Thus, by (1.65),(1.66) and the subadditivity of φ,(1.64) follows.

Since, by Theorem 1.92 (i), every ν-measurable set is also φ-measurable, φ is a Borel outer measure. Thus we have only to show that φ is a Borel regular outer measure, that is, given $C \subset X$, we must show that there exists a Borel set F such that $C \subset F$ and $\varphi(F) = \varphi(C)$.

Let $\varphi_i := \varphi \llcorner V_i = \nu \llcorner (E \cap V_i)$ (if $i \in \mathbb{N}$). Let us observe that

$$\exists\, \varphi(C) = \lim_{i\to\infty} \varphi_i(C) \quad \text{for all } C \subset X\,. \tag{1.67}$$

Indeed, by the continuity of outer measures on increasing sequences of sets and (1.63),

$$\lim_{i\to\infty} \varphi(C) = \lim_{i\to\infty} (\nu \llcorner (E \cap C))(V_i) = (\nu \llcorner (E \cap C))(X) = \varphi(C)\,.$$

On the other hand, since ν is Borel regular, $E \cap V_i$ is ν- measurable, and, by (1.64), $\nu(E \cap V_i) \leqslant \nu(V_i) < \infty$, by Theorem 1.92 (ii) we can infer that $\varphi_i = \nu \llcorner (E \cap V_i)$ is a Borel regular outer measure. Thus, for each i and for a given $C \subset X$, there exists a Borel set F_i such that

$$C \subset F_i \text{ and } \varphi_i(F_i) = \varphi_i(C)\,. \tag{1.68}$$

Let $F := \cap_{i=1}^{\infty} F_i$. Then F is still a Borel set with $F \supset C$ and

$$\varphi_i(C) \leqslant \varphi_i(F) \leqslant \varphi_i(F_i) = \varphi_i(C)\,.$$

Thus, it follws that there exists a Borel set $F \supset C$ such tha

$$\varphi_i(C) = \varphi_i(F) \quad \forall\, i\,.$$

By taking the limit, as $i \to \infty$, in the previous identity, we get that

$$\varphi(C) = \varphi(F)\,,$$

and, then, the desired conclusion. □

Given a measure space $(X, \mathcal{M})$ and a measure on it, we see now how it can be carried on another set Y through a function $f : X \to Y$.

Definition 1.95 (Push-forward of a measure, or image measure)**.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measure spaces, and let $f : X \to Y$ be *measurable*, that is $f^{-1}(F) \in \mathcal{M}$ whenever $F \in \mathcal{N}$. For any positive, real, or vector measure ν on $(X, \mathcal{M})$ we define a measure $f_{\#}\nu$ in $(Y, \mathcal{N})$ by

$$f_{\#}\nu(F) := \nu\left(f^{-1}(F)\right)$$

From the previous definition the corresponding change of variable formula for integrals follows immediately: if u is a (reaI- or vector-valued) function on Y integrable with respect to $f_{\#}\nu$, then $u \circ f$ in integrable with respect to ν, and we have the equality:

$$\int_Y u\, d(f_{\#}\nu) = \int_X u \circ f\, d\nu \tag{1.69}$$

The very general definition given above can be easily seen to have good properties in l.c.s. spaces when f is assumed to be continuous and proper, i.e. such that $f^{-1}(K)$ is compact for any compact $K \subset Y$ as the following remark shows.

Remark 1.96. Let X, Y be l.c.s. metric spaces, $f : X \to Y$ continuous and proper : the continuity of f ensures that $f^{-1}(B)$ whenever $B \in \mathcal{B}(Y)$, and since f is proper, if ν is a Radon measure on X, then $f_{\#}\nu$ is a Radon measure on Y.

Hausdorff measures provide an important source of examples of Radon measures. We will deeply study them in Chapter III. We are going now to stress some relationships between the 1-dimensional Hausdorff measure and the classical notion of length measure for a curve in $\mathbb{R}^n$.

Example 1.97 (Push-forward of the classical length measure)**.** A set $\Gamma \subset \mathbb{R}^n$ is a *curve* of $\mathbb{R}^n$ if there exists a continuous, injective function $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma([a, b]) = \Gamma$. The function γ is called a *parametrization* of Γ. Given a parametrization $\gamma : [a, b] \to \mathbb{R}^n$ and a subinterval $[c, d] \subseteq [a, b]$, we define the *length of* γ *over* $[c, d]$ as

$$\operatorname{length}(\gamma; [c, d]) := \sup \left\{ \sum_{i=1}^{N} |\gamma(t_i) - \gamma(t_{i-1}| : t_0 = c < t_1 < \cdots < t_N = d \right\} \tag{1.70}$$

where the supremum is taken over all finite partitions $\{t_0 = c < t_1 < \cdots < t_N = d\}$ of $[c, d]$. It can be proved that, if $\Gamma = \gamma_1([a_1, b_1]) = \gamma_2([a_2, b_2])$ for two given parametrizations $\gamma_i : [a_i, b_i] \to \mathbb{R}^n$ $(i = 1, 2)$, then $\operatorname{length}(\gamma_1; [a_1, b_1]) = \operatorname{length}(\gamma_2; [a_2, b_2])$. Thus we can define as *length of* $\Gamma = \gamma([a, b])$ the quantity

$$\operatorname{length}(\Gamma) := \operatorname{length}(\gamma; [a, b]) .$$

It is also well-known that, if Γ is a $\mathbf{C}^1$ regular curve, that is there exists a $\mathbf{C}^1$ parametrization $\gamma : [a, b] \to \mathbb{R}^n$ of Γ with $|\gamma'(t)| \neq 0$ for each $t \in [a, b]$, then

$$\operatorname{length}(\gamma; [c, d]) = \int_c^d |\gamma'(t)|\, dt \quad \forall [c, d] \subset [a, b] . \tag{1.71}$$

Let us recall that the 1-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^1(E) := \lim_{\delta \to 0} \mathcal{H}^1_\delta(E) = \sup_{\delta \in (0,\infty)} \mathcal{H}^1_\delta(E) ,$$

where

$$\mathcal{H}^1_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(E_i) : \ E \subset \bigcup_{i=1}^{\infty} E_i, \quad \operatorname{diam}(E_i) \leqslant \delta\right\}.$$

Given a curve Γ, it holds that

$$\mathcal{H}^1(\Gamma) = \operatorname{length}(\Gamma)\,, \tag{1.72}$$

whether length(Γ) is finite or not (see Theorem 3.25).

Assume now $\Gamma = \gamma([a, b])$ is $\mathbf{C}^1$ regular curve and define the measures

$$\nu(E) := \mathcal{H}^1 \llcorner \Gamma(E) := \mathcal{H}^1(\Gamma \cap E) \text{ if } E \in \mathcal{B}(\mathbb{R}^n)\,,$$

$$\mu(E) := \int_E |\gamma'(t)|\, dt \text{ if } E \in \mathcal{B}([a, b])\,.$$

Then, by (1.71) and (1.72), it follows that ν and μ are finite Radon measure respectively on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and $([a, b], \mathcal{B}([a, b])$. Moreover, according to Definition 1.95,

$$\nu = \gamma_\# \mu\,. \tag{1.73}$$

In particular, by (1.73) and (1.69), it follows that

$$\int_\Gamma \varphi\, d\mathcal{H}^1 = \int_{\mathbb{R}^n} \varphi\, d\nu = \int_a^b \varphi \circ \gamma\, d\mu = \int_a^b \varphi(\gamma(t))\, |\gamma'(t)|\, dt \tag{1.74}$$

for each $\varphi \in \mathbf{C}^0_c(\mathbb{R}^n)$.

We consider now two measure spaces and see the resulting structure on their cartesian product. In particular we introduce Fubini and Tonelli theorems which generalizes the notion of iterated integration of Riemannian calculus.

Definition 1.98 (Product σ-algebra)**.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measure spaces. The *product σ-algebra of $\mathcal{M}$ and $\mathcal{N}$* denoted by $\mathcal{M} \times \mathcal{N}$ is the σ-algebra generated in $X \times Y$ by

$$\mathcal{G} = \{E \times F : \ E \in \mathcal{M}, F \in \mathcal{N}\}\ .$$

Remark 1.99. Let $S \in \mathcal{M} \times \mathcal{N}$; then for every $x \in X$ the section $S_x := \{y \in Y : (x, y) \in S\}$ belongs to $\mathcal{N}$ and for every $y \in Y$ the section $S_y := \{x \in X : \ (x, y) \in S\}$ belongs to $\mathcal{M}$. In fact it is easily checked that the collections

$$\mathcal{S}_X := \{S \in \mathcal{M} \times \mathcal{N} : \ S_x \in \mathcal{N}, \forall x \in X\}\,, \mathcal{S}_Y := \{S \in \mathcal{M} \times \mathcal{N} : \ S_y \in \mathcal{M}, \forall y \in Y\}$$

are σ-algebras in $X \times Y$ and contain $\mathcal{G}$. Thus $\mathcal{S}_X = \mathcal{S}_Y = \mathcal{M} \times \mathcal{N}$ (see [R1, Theorem 8.2]).

Remark 1.100. If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are complete measure spaces (see Remark 1.6), it need not be true that the product algebra $\mathcal{M} \times \mathcal{N}$ is complete. Indeed, suppose there exists $E \in \mathcal{M}$, $E \neq \emptyset$ with $\mu(E) = 0$; and suppose there exists $F \subset Y$ such that $F \notin \mathcal{N}$. Then $E \times F \subset E \times Y$, $(\mu \times \nu)(E \times Y) = 0$, but, by Remark 1.99, $E \times F \notin \mathcal{M} \times \mathcal{N}$. Therefore we will consider the completion $(\mathcal{M} \times \mathcal{N})^*$ in place of $\mathcal{M} \times \mathcal{N}$.

Theorem 1.101 (Fubini)**.** *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are complete σ-finite measure spaces.*

(i) *There exists a unique σ-finite measure on $(X \times Y, (\mathcal{M} \times \mathcal{N})^*)$, denoted with $\mu \times \nu$ and called* **product measure** *between μ and ν, such that*

$$(\mu \times \nu)(E \times F) = \mu(E)\, \nu(F) \quad \forall\, E \in \mathcal{M},\ F \in \mathcal{N}\,,$$

(where we define $0\,\infty = \infty\, 0 = 0$).

(ii) *If $S \in (M_1 \times \mathcal{N})^*$ then*

$$S_y := \{x \in X : (x, y) \in X \times Y\} \in \mathcal{N} \quad \textit{for } \nu\textit{-a.e. } y \in Y\,,$$

$$S_x := \{y \in Y : (x, y) \in X \times Y\} \in \mathcal{N} \quad \textit{for } \mu\textit{-a.e. } x \in X\,,$$

$$y \mapsto \mu(S_y) \ \textit{is } \mathcal{N}\textit{-measurable}, x \mapsto \mu(S_x) \ \textit{is } \mathcal{M}\textit{-measurable}\,,$$

$$\begin{aligned}(\mu \times \nu)(S) &= \int_X \nu(S_x)\, d\mu(x) = \int_X \left(\int_Y \chi_S(x, y)\, d\nu(y)\right) d\mu(x) \\ &= \int_Y \mu(S_y)\, d\nu(y) = \int_Y \left(\int_X \chi_S(x, y)\, d\mu(x)\right) d\nu(y)\,.\end{aligned}$$

(iii) *If $u \in L^1(X \times Y, (\mathcal{M} \times \mathcal{N})^*, \mu \times \nu)$ then*

$$y \mapsto u(x, y) \ \textit{is } \nu\textit{-integrable for } \mu\textit{-a.e. } x \in X,$$

$$x \mapsto u(x, y) \ \textit{is } \mu\textit{-integrable for } \nu\textit{-a.e. } y \in Y,$$

$$\begin{aligned}\int_{X \times Y} u\, d(\mu \times \nu) &= \int_X \left(\int_Y u(x, y)\, d\nu(y)\right) d\mu(x) \\ &= \int_Y \left(\int_X u(x, y)\, d\mu(x)\right) d\nu(y)\,.\end{aligned}$$

Theorem 1.102 (Tonelli)**.** *Under the same assumptions of Theorem 1.101, if $f : X \times Y \to [0, \infty]$ is $(\mathcal{M} \times \mathcal{N})^*$-measurable, then*

$$y \mapsto u(x, y) \ \textit{is } \mathcal{N}\textit{-measurable for } \mu\textit{-a.e. } x \in X\,,$$

$$x \mapsto u(x, y) \ \textit{is } \mathcal{M}\textit{-measurable for } \nu\textit{-a.e. } y \in Y\,,$$

$$x \mapsto \int_Y u(x, y)\, d\nu(y) \ \textit{is } \mathcal{M}\textit{-measurable},$$

$$y \mapsto \int_X u(x, y)\, d\mu(x) \ \textit{is } \mathcal{N}\textit{-measurable},$$

and

$$\int_{X \times Y} u\, d(\mu \times \nu) = \int_X \left(\int_Y u(x, y)\, d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X u(x, y)\, d\mu(x)\right) d\nu(y)$$

in the sense that either both expressions are infinite or both are finite and equal.

For the proofs of Theorems 1.101 and 1.102 we refer to [R1, Theorems 8.8 and 8.12] or [GZ, Theorem 6.46 and Corollary 6.47].

Example 1.103 (Counterexample to Fubini-Tonelli's theorem)**.** Fubini and Tonelli's theorems may fail when μ or ν is not σ-finite. Let $X_1 = X_2 = [0,1]$, $\mathcal{M} := \mathcal{M}_1 \cap [0,1]$ and $\mathcal{N} := \mathcal{P}([0,1])$ be respectively the class of Lebesgue measurable sets and of all subsets in $[0,1]$, $\mu = \mathcal{L}^1 \llcorner [0,1]$ and $\nu = \#$ be respectively the Lebesgue measure and the counting measure in $[0,1]$. Let $u : [0,1]^2 \to [0,\infty)$, $u(x,y) := \chi_D(x,y)$, where $D := \{(x,y) \in [0,1]^2 : x = y\}$. Then u is $(\mathcal{M} \times \mathcal{N})$-measurable, since $D \in \mathcal{M} \times \mathcal{N}$. Indeed $D = \cap_{h=1}^{\infty} Q_h$ with $Q_h := \cup_{i=1}^{h} [(i-1)/h, i/h]^2 \in \mathcal{M} \times \mathcal{N}$. On the other hand

$$\int_X u(x,y)\, d\mu(x) = 0 \,\forall y \in [0,1], \qquad \int_Y u(x,y)\, d\nu(y) = 1 \,\forall x \in [0,1],$$

so that

$$\int_Y \left(\int_X u(x,y)\, d\mu(x) \right) d\nu(y) = 0, \qquad \int_X \left(\int_Y u(x,y)\, d\nu(y) \right) = 1\,.$$

The failure of Theorems 1.101 (ii) and 1.102 is due to the fact that ν is not σ-additive.

An interesting application of Fubini and Tonelli's theorems is the following Cavalieri's principle.

Proposition 1.104 (Cavalieri's principle)**.** *Let $(X, \mathcal{M})$ be a measure space, μ a positive measure on it and $u : X \to [0,\infty]$ be measurable. Let $[0,\infty) \ni t \mapsto \mu(\{u > t\})$ denote the* ***distribution function of*** *u, that is,*

$$\mu(\{u > t\} = \mu\left(\{x \in X : u(x) > t\}\right) \text{ if } t \in [0,\infty)\,. \tag{1.75}$$

Let $\theta : [0,\infty) \to [0,\infty)$ be (strictly) increasing such that $\theta(0) = 0$, $\theta : [0,T] \to [0,\infty)$ is absolutely continuous for each $T \in [0,\infty)$. Then

$$\int_X (\theta \circ u)\, d\mu = \int_0^\infty \theta'(t)\, \mu(\{u > t\})\, dt\,. \tag{1.76}$$

In particular, if $\theta(t) = t^p$ with $p \geqslant 1$, then

$$\int_X u^p\, d\mu = p \int_0^\infty t^{p-1} \mu(\{u > t\})\, dt\,.$$

Proof. Let $S := \{x \in X : u(x) > 0\}$ and consider the restriction measure of μ to S $\mu^* := \mu \llcorner S$.

1st step: Assume that μ^* is not σ-finite. Then it is easy to see that both sides are ∞. Indeed, since $S := \cup_{h=1}^{\infty} \{x \in X : u(x) > 1/h\}$, there exists an integer $\bar{h}$ such that $\mu(\{u > 1/\bar{h}\} = \infty$. Thus

$$\begin{aligned}\int_X \theta \circ u\, d\mu &\geqslant \int_{\{u > 1/\bar{h}\}} \theta \circ u\, d\mu \geqslant \theta\left(\frac{1}{\bar{h}}\right) \mu(\{u > 1/\bar{h}\}) = \infty \\ &= \mu(\{u > 1/\bar{h}\} \int_0^{1/\bar{h}} \theta'(t)\, dt \leqslant \int_0^\infty \theta'(t)\, \mu(\{u > t\})\, dt\,.\end{aligned}$$

2nd step: Assume that μ^* is σ-finite. Then, since

$$\int_X \theta \circ u\, d\mu = \int_X \theta \circ u\, d\mu^* \text{ and } \int_0^\infty \theta'(t)\, \mu(\{u > t\}\, dt = \int_0^\infty \theta'(t)\, \mu^*(\{u > t\})\, dt\,,$$

we can assume that μ itself is σ-finite. Let

$$E := \{(x,t) \in X \times [0,\infty) : u(x) > t\}\,.$$

$$E_t := \{x \in X : u(x) > t\} \text{ if } t \in [0, \infty).$$

Then it can be proved that $E \in \mathcal{M} \times \mathcal{B}([0,\infty))$ and the distribution function of u is then

$$\mu(E_t) = \int_X \chi_E(x,t)\, d\mu(x) \quad \forall t \in [0,\infty).$$

Therefore, the right side of (1.76) is, by applying Fubini's theorem with $\mu \times \mathcal{L}^1$,

$$\begin{aligned} \int_0^\infty \mu(E_t)\, \theta'(t)\, dt &= \int_0^\infty \left(\int_X \chi_E(x,t)\, \theta'(t)\, d\mu(x) \right) dt \\ &= \int_X \left(\int_0^\infty \chi_E(x,t)\, \theta'(t)\, dt \right) d\mu(x). \end{aligned} \tag{1.77}$$

For a given $x \in X$, observe that $\theta : [0,T] \to \mathbb{R}$ is absolutely continuous for each $0 \leqslant T < u(x) \leqslant \infty$. Thus it follows that

$$\begin{aligned} \int_0^\infty \chi_E(x,t)\, \theta'(t)\, dt &= \int_0^{u(x)} \theta'(t)\, dt = \lim_{T \to u(x)^-} \int_0^T \theta'(t)\, dt \\ &= \lim_{T \to u(x)^-} \theta(T) = \theta(u(x)). \end{aligned} \tag{1.78}$$

From (1.77) and (1.78), (1.76) follows. □

1.7. **Weak*-convergence of measures. Regularization of Radon measures in $\mathbb{R}^n$.**

Weak*-convergence of measures.

In this section we will assume that (X,d) is a l.c.s. metric space, the characterization of the spaces of measures $(\mathcal{M}_{\text{loc}}(X))^m$ and $(\mathcal{M}(X))^m$ (see Corollary 1.78 and Theorem 1.83) as dual spaces induce on them a natural notion of weak*-convergence.

Definition 1.105. Let (X,d) be a l.c.s. metric space.

(i) Let $(\nu_h)_h$ and ν be measures in $(\mathcal{M}_{\text{loc}}(X))^m$; we say that $(\nu_h)_h$ *locally weakly* converges* to ν, and write $\nu_h \overset{*}{\rightharpoonup} \nu$, if

$$\lim_{h\to\infty} \int_X u\, d\nu_h = \int_X u\, d\nu \quad \forall u \in \mathbf{C}_c^0(X).$$

(ii) Let $(\nu_h)_h$ and ν be measures in $(\mathcal{M}(X))^m$; we say that $(\nu_h)_h$ *weakly* converges* to ν if

$$\lim_{h\to\infty} \int_X u\, d\nu_h = \int_X u\, d\nu \quad \forall u \in \mathbf{C}_0^0(X).$$

Remark 1.106. The weak*-convergence of finite Radon measures is called *vague convergence*, mainly in probability, and it is of considerable importance in applications.

Remark 1.107. Weak* convergence of $(\nu_h)_h \subset \mathcal{M}(X)$ to $\nu \in \mathcal{M}(X)$ does not imply that $\lim_{h\to\infty} \nu_h(A) = \nu(A)$, even when $A = X$ as the following exercise shows.

Exercise: Let $X = \mathbb{R}$ and let $\nu_h := \delta_h$. Then prove that $(\nu_h)_h$ weakly* converges to $\nu \equiv 0$, but $\lim_{h\to\infty} \nu_h(\mathbb{R}) = 1 \neq 0 = \nu(\mathbb{R})$.

Remark 1.108. The local weak* convergence is unique. More precisely

Exercise: Given $(\nu_h)_h \subset (\mathcal{M}_{loc}(X))^m$ and $\nu_i \in (\mathcal{M}(X))^m$ $i = 1,2$. If $\nu_h \overset{*}{\rightharpoonup} \nu_i$ as $h \to \infty$ for $i = 1,2$, then $\nu_1(E) = \nu_2(E)$ for each $E \in \mathcal{B}_{\text{comp}}(X)$.
(*Hint:* If $\nu_h := \left(\nu_h^{(1)}, \ldots, \nu_h^{(m)}\right)$, $\nu_i := \left(\nu_i^{(1)}, \ldots, \nu_i^{(m)}\right) : \mathcal{B}_{\text{comp}}(X) \to \mathbb{R}^m$, let define the functionals L_h, $L_i : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$, if $\psi = (\psi_1, \ldots, \psi_m)$,

$$L_h(\psi) := \int_X \sum_{j=1}^m \psi_j \, d\nu_h^{(j)}, \quad L_i(\psi) := \int_X \sum_{j=1}^m \psi_j \, d\nu_i^{(j)} .$$

Prove that:

- both L_h and L_i are linear continuous functionals according to Definition 1.69;
- $L_1(\psi) = L_2(\psi)$ for all $\psi \in (\mathbf{C}_c^0(X))^m$.

Applying Riesz representation theorem 1.73, conclude that $\nu_1 = \nu_2$.)

Proposition 1.109 (Locally weak* convergence vs. weak* convergence)**.** *Assume that* $(\nu_h)_h$, $\nu \subset \mathcal{M}_{\text{loc}}(X)$. *Then they are equivalent:*

(i) $\nu_h \overset{*}{\rightharpoonup} \nu$ *and* $\sup_h |\nu_h|(X) < \infty$*;*
(ii) $(\nu_h)_h$, $\nu \subset \mathcal{M}(X)$ *and* $(\nu_h)_h$ *weakly* converges to* ν.

.

Proof. (ii)⇒ (i): it is trivial, by definition, that $\nu_h \overset{*}{\rightharpoonup} \nu$. By the characterization of $\mathcal{M}(X)$ (see Theorem 1.83)

$$|\nu_h|(X) = \|\nu_h\| = \|L_{\nu_h}\|_{(\mathbf{C}_0^0(X), \|\cdot\|_\infty)'} \tag{1.79}$$

and

$$(\nu_h)_h \text{ weakly* converges to } \nu \iff (L_{\nu_h})_h \text{ weakly* converges to } L_\nu \text{ in } (\mathbf{C}_0^0(X), \|\cdot\|_\infty)' . \tag{1.80}$$

Thus, by (1.80),$(L_{\nu_h})_h$ is bounded in $(\mathbf{C}_0^0(X), \|\cdot\|_\infty)'$ and, by (1.79), it follows that $\sup_h |\nu_h|(X) < \infty$.

(i)⇒ (ii): By arguing as in the previuos implication, we have that the sequence $(L_{\nu_h})_h$ is bounded in $(\mathbf{C}_0^0(X), \|\cdot\|_\infty)'$. By the sequential weak*-compactness of bounded sets of $(\mathbf{C}_0^0(X), \|\cdot\|_\infty)'$ and Theorem 1.83, we have that, up to a subsequence,

$$(L_{\nu_h})_h \text{ weakly* converges to } L_{\nu^*} \text{ in } (\mathbf{C}_0^0(X), \|\cdot\|_\infty)' . \tag{1.81}$$

for some $\nu^* \in \mathcal{M}(X)$. By the assumptions and (1.81), we can infer that

$$L_\nu(u) = L_{\nu^*}(u) \quad \forall\, u \in \mathbf{C}_c^0(X) .$$

Since $\mathbf{C}_c^0(X)$ is dense in $(\mathbf{C}_0^0(X), \|\cdot\|_\infty)$, by the previous identity, we get that $\nu = \nu^*$ and then the desired conclusion. □

Remark 1.110. The weak*-convergence of Radon measures if stable with respect to the push-forward of Radon measures. Indeed, let X, Y be l.c.s. metric spaces, $f : X \to Y$ continuous and proper: the continuity of f ensures that $f^{-1}(B)$ whenever $B \in \mathcal{B}(Y)$, and since f is proper the spaces $\mathbf{C}_c^0(Y)$ and $\mathbf{C}_0^0(Y)$ are continuously mapped in $\mathbf{C}_c^0(X)$ and $\mathbf{C}_0^0(X)$ respectively by $u \mapsto u \circ f$. Thus, if the sequence $(\nu_h)_h$ of Radon measures on X locally weakly* converges to the measure ν, then the sequence $(f_\# \nu_h)_h$ locally weakly* converges to $f_\# \nu$, and the same statement holds for finite Radon measures and weak*-convergence.

Example 1.111 (Blow-ups of a curve in $\mathbb{R}^n$)**.** A fundamental idea in GMT is the existence of tangent spaces to irregular submanifolds in terms of weak*-convergence of suitable Radon measures. This idea will be developed later and we now sketch it with an example. Let Γ be a $\mathbf{C}^1$ regular curve of $\mathbb{R}^n$, that is $\Gamma = \gamma([a,b])$ for a $\mathbf{C}^1$ curve $\gamma : [a,b] \to \mathbb{R}^n$, injective with $|\gamma'(t)| \neq 0$ for each $t \in [a,b]$. Given $t_0 \in (a,b)$, the tangent space to Γ at $x_0 = \gamma(t_0)$ is the line $\pi = \{s\, \gamma'(t_0) : s \in \mathbb{R}\}$. Consider now Γ as a Radon measure, looking at $\nu = \mathcal{H}^1 \llcorner \Gamma$, and define the *blow-ups* $\nu_{x_0,r}$ of ν at x_0, setting

$$\nu_{x_0,r} := \frac{1}{r} (\Phi_{x_0,r})_\# (\mathcal{H}^1 \llcorner \Gamma)\,,$$

with $\Phi_{x_0,r}(y) := \dfrac{y - x_0}{r}$ if $y \in \mathbb{R}^n$.

Exercise: Prove that

$$\nu_{x_o,r}(E) = \mathcal{H}^1 \llcorner \left(\frac{\Gamma - x_0}{r} \right)(E) \quad \forall\, E \in \mathcal{B}(\mathbb{R}^n)\,.$$

(*Hint.* Use (1.69) and (1.73).)

Let us check that the property of π to be the tangent space implies that $\nu_{x_0,r} \overset{*}{\rightharpoonup} \mathcal{H}^1 \llcorner \pi$. Indeed, by (1.74), if $\varphi \in \mathbf{C}^0_c(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi \, d\nu_{x_0,r} &= \frac{1}{r} \int_\Gamma \varphi \left(\frac{y - x_0}{r} \right) d\mathcal{H}^1(y) = \frac{1}{r} \int_a^b \varphi \left(\frac{\gamma(t) - \gamma(t_0)}{r} \right) |\gamma'(t)|\, dt \\
&= \int_{-(t_0 - a)/r}^{(b - t_0)/r} \varphi \left(\frac{\gamma(t_0 + rs) - \gamma(t_0)}{r} \right) |\gamma'(t_0 + rs)|\, ds \\
&\to \int_{\mathbb{R}} \varphi\,(s\, \gamma'(t_0))\, |\gamma'(t_0)|\, ds = \int_\pi \varphi \, d\mathcal{H}^1\,, \text{as } r \to 0^+\,.
\end{aligned}$$

Other interesting examples of weak*-converging sequences of measures illustrating a wide variety of behaviours can be found in [Mag, Examples 4.20-4.23].

We now characterize the local weak*-convergence of positive Radon measures in terms of evaluation on sets and introduce a criterion about the *narrow convergence* of positive finite Radon measures.

Let us first introduce an useful criterion about foliations of Borel sets.

Lemma 1.112 (Foliations of Borel sets for positive Radon measures)**.** *Let (X,d) be a l.c.s. metric space. If $\{E_t\}_{t \in I}$ is a disjoint family of Borel sets in X, indexed over some set I, and μ is a positive Radon measure on $(X, \mathcal{B}(X))$. Then $\mu(E_t) > 0$ for at most countably many $t \in I$.*

Proof. By Lemma 1.17, we can assume there exists an increasing sequence of compact sets $(K_h)_h$ of (X,d) such that $X = \cup_{h=1}^\infty K_h$. Let $I_h := \{t \in I : \mu(E_t \cap K_h) > 1/h\}$. Observe that

$$\{t \in I : \mu(E_t) > 0\} = \cup_{h=1}^\infty I_h\,. \tag{1.82}$$

Indeed, it is trivial that $\cup_{h=1}^{\infty} I_h \subset \{t \in I : \mu(E_t) > 0\}$. Let us then prove the reverse inclusion. Assume that $\mu(E_t) > 0$ for some $t \in I$. Then there exists an integer h_0 such that $\mu(E_t) > 1/h_0$. On the other hand, since $\lim_{h\to\infty} \mu(E_t \cap K_h) = \mu(E_t)$, there exists an integer $h > h_0$ such that $\mu(E_t \cap K_h) > 1/h_0 > 1/h$. Thus $t \in I_h$ and (1.82) follows.

Let us now show that I_h is finite with $\#(I_h) \leqslant h\mu(K_h)$. Therefore, by (1.82), the proof will be accomplished. Let $J \subset I_h$ be finite. Then

$$\mu(K_h) \geqslant \mu\left(\cup_{t\in I_h}(E_t \cap K_h)\right) \geqslant \mu\left(\cup_{t\in J}(E_t \cap K_h)\right) = \sum_{t\in J} \mu((E_t \cap K_h)) \geqslant \frac{\#(J)}{h}.$$

□

Theorem 1.113 (Characterization of the locally weak* convergence of positive Radon measures). *Let $(\mu_h)_h$ and μ be positive Radon measures on $(X, \mathcal{B}(X))$. Then the following are equivalent.*

(i) $\mu_h \overset{*}{\rightharpoonup} \mu$ *as* $h \to \infty$.

(ii) *If K compact and A open, then*

$$\mu(K) \geqslant \limsup_{h\to\infty} \mu_h(K)\,, \tag{1.83}$$

$$\mu(A) \leqslant \liminf_{h\to\infty} \mu_h(A)\,. \tag{1.84}$$

(iii) *If $E \in \mathcal{B}_{\mathrm{comp}}(X)$ with $\mu(\partial E) = 0$, then*

$$\mu(E) = \lim_{h\to\infty} \mu_h(E)\,.$$

Moreover, if $\mu_h \overset{*}{\rightharpoonup} \mu$ as $\to \infty$, then for every $x \in \mathrm{spt}\mu$ there exists $(x_h)_h \subset X$ with

$$\lim_{h\to\infty} x_h = x, \quad x_h \in \mathrm{spt}\mu_h \quad \forall\, h \in \mathbb{N}\,. \tag{1.85}$$

Proof. <u>*(i) ⇒ (ii)*</u>: Let K' and A' be respectively a compact and open set such that $K' \subset A'$. By Urysohn's lemma 1.65 there exists a function $\varphi \in \mathbf{C}^0_c(X)$ such that $\chi_{K'} \leqslant \varphi \leqslant \chi_{A'}$, then

$$\mu_h(K') \leqslant \int_X \varphi\, d\mu_h \leqslant \mu_h(A')\,\forall\, h, \quad \mu(K') \leqslant \int_X \varphi\, d\mu \leqslant \mu(A')\,.$$

By (i) and the previous inequality we have

$$\limsup_{h\to\infty} \mu_h(K') \leqslant \int_X \varphi\, d\mu \leqslant \mu(A')\,, \tag{1.86}$$

$$\liminf_{h\to\infty} \mu_h(A') \geqslant \int_X \varphi\, d\mu \geqslant \mu(K')\,. \tag{1.87}$$

Setting $K' = K$ in (1.86) and passing to the infimum on all open sets $A' \supset K$, by Theorem 1.14 (i), (1.83) follows. Setting $A' = A$ in (1.87) and passing to the supremum on all compact sets $K' \subset A$, by Theorem 1.14 (ii), (1.84) follows.

<u>*(ii) ⇒ (iii)*</u>: Notice that

$$\mu(\overset{\circ}{E}) \leqslant \mu(E) \leqslant \mu(\bar{E}) = \mu(\overset{\circ}{E}) + \mu(\partial E) = \mu(\overset{\circ}{E})\,,$$

then, since $\bar{E}$ is compact, combining (ii) and the monotonicity of μ

$$\mu(E) = \mu(\overset{\circ}{E}) \leqslant \liminf_{h\to\infty} \mu_h(\overset{\circ}{E}) \leqslant \liminf_{h\to\infty} \mu_h(E)$$
$$\leqslant \limsup_{h\to\infty} \mu_h(E) \leqslant \limsup_{h\to\infty} \mu_h(\bar{E}) \leqslant \mu(\bar{E}) = \mu(E)\,.$$

In particular it follows that $\mu(E) = \liminf_{h\to\infty} \mu_h(E) = \limsup_{h\to\infty} \mu_h(E)$, and then the desired conclusion.

(iii) ⇒ (i): Let $\varphi \in \mathbf{C}^0_c(X)$, $\varphi \geqslant 0$. By Lemma 1.112, there exists $I \subset [0,\infty)$ such that $\mathcal{L}^1([0,\infty)\setminus I) = 0$

$$\mu(\{\varphi = t\}) = 0 \quad \forall t \in I\,.$$

Being φ continuous,

$$\partial\{\varphi > t\} \subseteq \{\varphi = t\} \quad \forall t \in [0,\infty)\,.$$

Hence (iii) implies that

$$\mu(\{\varphi > t\}) = \lim_{h\to\infty} \mu_h(\{\varphi > t\}) \quad \forall t \in I\,.$$

Let $f_h, f : [0,\infty) \to \mathbb{R}$ $(h = 1,2,\dots)$, $f_h(t) := \mu_h(\{\varphi > t\})$, and $f(t) := \mu(\{\varphi > t\})$. Then, being f_h and f nondecreasing functions, f_h and f are Borel measurable and

$$\lim_{h\to\infty} f_h(t) = f(t) \quad \mathcal{L}^1\text{-a.e. } t \in [0,\infty]\,,$$

$$|f_h(t)| \leqslant \mu(\mathrm{spt}\varphi)\, \chi_{[0,\sup_{\mathbb{R}^n}\varphi]}(t) \quad \forall t \in [0,\infty)\,.$$

By dominated convergence theorem and Cavalieri's principle (1.76) with $\theta(t) = t$, we have

$$\int_X \varphi\, d\mu = \int_0^\infty \mu(\{\varphi > t\})\, dt = \lim_{h\to\infty} \int_0^\infty \mu_h(\{\varphi > t\})\, dt = \lim_{h\to\infty} \int_X \varphi\, d\mu_h\,.$$

If φ has a general sign, we can decompose it as $\varphi = \varphi_+ - \varphi_-$ and apply the previous argument to φ_+ and φ_-.

We finally prove (1.85) and let us prove that for every $\epsilon > 0$ there exists $\bar{h} \in \mathbb{N}$ such that $\mathrm{spt}\mu_h \cap B(x,\epsilon) \neq \emptyset$. By contradiction, there exists $\epsilon_0 > 0$ and an increasing sequence of integers $(h_k)_h$ with $\lim_{k\to\infty} h_k = \infty$ such that $\mathrm{spt}\mu_{h_k} \cap B(x,\epsilon_0) = \emptyset$ for each k. By (1.84), it follows that

$$\mu(B(x,\epsilon_0)) \leqslant \liminf_{k\to\infty} \mu_{h_k}(B(x,\epsilon_0)) = 0\,,$$

which contradicts the fact that $x \in \mathrm{spt}\mu$ and then $\mu(B(x,\epsilon_0)) > 0$. □

Remark 1.114. *(*Limits points of support points and uniform lower bounds [Mag, Remark 4.28]*)* If $\mu_h \overset{*}{\rightharpoonup} \mu$, $x_h \in \mathrm{spt}\mu_h$ for every $h \in \mathbb{N}$, and $x_h \to x$, then it is not true, in general, that $x \in \mathrm{spt}\mu$. For instance, if $X = \mathbb{R}$, the sequences

$$\mu_h = \left(1 - \frac{1}{h}\right)\delta_1 + \frac{1}{h}\,\delta_{1/h}, \quad x_h = \frac{1}{h}\,.$$

The implication becomes true as soon as some kind of uniform lower bound on the measure assigned by the μ_h around their support points is assumed. More precisely, let $(\mu_h)_h$ be a sequence of positive Radon measures on X, such that, for every $r > 0$,

(1.88) $$\limsup_{h\to\infty} \inf\{\mu_h(B(x,r)) : x \in \mathrm{spt}\mu_h\} > 0\,.$$

Under this assumption, we claim that, if $\mu_h \overset{*}{\rightharpoonup} \mu$, $x_h \to x$, and $x_h \in \mathrm{spt}\mu_h$, then $x \in \mathrm{spt}\mu$. Indeed, let $c(r)$ denote the left-hand side of (1.88). For every $r > 0$, let $h_0 \in \mathbb{N}$ be such that $B(x_h, r) \subset B(x, 2r)$ for every $h \geqslant h_0$. By (1.83), and if necessary extracting a subsequence so as to exploit (1.88),

$$\mu(B(x,2r)) \geqslant \limsup_{h\to\infty} \mu_h(B(x,2r)) \geqslant \limsup_{h\to\infty} \mu_h(B(x_h,r)) \geqslant c(r) > 0 .$$

By the arbitrariness of r, we find that $x \in \mathrm{spt}\mu$.

Proposition 1.115 (Characterization of the narrow convergence of positive Radon measures)**.** *Let* $(\mu_h)_h$ *be a sequence of positive, finite Radon measures on* $(X, \mathcal{B}(X))$ *and assume the existence of a positive, finite Radon measure* μ *such that*

$$\lim_{h\to\infty} \mu_h(X) = \mu(X) \ \textit{and} \ \liminf_{h\to\infty} \mu_h(A) \geqslant \mu(A) \tag{1.89}$$

for every $A \subset X$ *open set. Then*

$$\lim_{h\to\infty} \int_X u\, d\mu_h = \int_X u\, d\mu \quad \forall\, u \in \mathbf{C}^0_b(X) \tag{NC}$$

where $\mathbf{C}^0_b(X)$ *denotes the class of all bounded continuous function* $u : X \to \mathbb{R}$*. In particular* $(\mu_h)_h$ *weakly* converges to* μ*. Moreover if* (NC) *holds so does* (1.89)*, that is* (NC) *and* (1.89) *are equivalent.*

Proof. Let $u \in \mathbf{C}^0_b(X)$. Possibly replacing u by $\alpha u + \beta$ for suitable α, $\beta \in \mathbb{R}$, we can assume without loss of generality that $0 \leqslant u \leqslant 1$. We first show that

$$\liminf_{h\to\infty} \int_X v\, d\mu_h \geqslant \int_X v\, d\mu , \tag{1.90}$$

for each continuous function $v : X \to [0, \infty)$. Indeed, by Cavalieri's principle (1.76) with $\theta(t) = t$ and Fatou lemma, we infer

$$\begin{aligned} \liminf_{h\to\infty} \int_X v\, d\mu_h &= \liminf_{h\to\infty} \int_0^\infty \mu_h(\{v > t\})\, dt \geqslant \int_0^\infty \liminf_{h\to\infty} \mu_h(\{v > t\})\, dt \\ &\geqslant \int_0^\infty \mu(\{v > t\})\, dt . \end{aligned}$$

Let us recall the following

Exercise: Prove that, if (a_h) and $(b_h)_h$ are sequences of real numbers such that

$$a \leqslant \liminf_{h\to\infty} a_h,\ b \leqslant \liminf_{h\to\infty} b_h,\ \limsup_{h\to\infty}(a_h + b_h) \leqslant a + b ,$$

for some a, $b \in \mathbb{R}$, then there exist $\lim_{h\to\infty} a_h = a$ and $\lim_{h\to\infty} b_h = b$.

Setting

$$a_h = \int_X u\, d\mu_h,\ a = \int_X u\, d\mu,\ b_h = \int_X (1-u)\, d\mu_h,\ b = \int_X (1-u)\, d\mu ,$$

from (1.90) and the assumption $\lim_{h\to\infty} \mu_h(X) = \mu(X)$, the proof is accomplished.

Finally, if (NC) holds, by choosing as test function $\varphi \equiv 1$ in (NC), we get that

$$\mu(X) = \lim_{h\to\infty} \mu_h(X) .$$

On the other hand, since (NC) also implies the local weak* convergence of $(\mu_h)_h$ to μ, by Theorem 1.113, it follows that, for each open set $A \subset X$,

$$\liminf_{h\to\infty} \mu_h(A) \geqslant \mu(A)\,.$$

Thus (1.89) follows. □

Remark 1.116. Convergence (NC) is called *narrow convergence of* $(\mu_h)_h$ *to* μ, and sometimes (by abuse of notation) *weak-convergence of* $(\mu_h)_h$ *to* μ. It is a stronger convergence than the weak*-convergence: indeed, for instance, $\lim_{h\to\infty} \mu_h(X) = \mu(X)$ is not granted for the weak*-convergence (recall the previous exercise). Note also that

$$\mathbf{C}^0_c(X) \subset \mathbf{C}^0_0(X) \subset \mathbf{C}^0_b(X)\,,$$

and $\mathbf{C}^0_c(X) = \mathbf{C}^0_0(X) = \mathbf{C}^0_b(X) = \mathbf{C}^0(X)$ if X is compact, and in this case the two notions of convergence coincide. On the other hand, for non-compact X the space $\mathcal{M}(X)$ is not the dual of $(\mathbf{C}^0_b(X), \|\cdot\|_\infty)$ as the following exercise shows.

Exercise: Let $X = \mathbb{R}$ and let $\mathcal{F} := \{f \in \mathbf{C}^0(\mathbb{R}) : \exists\, f(\infty) := \lim_{|x|\to\infty} f(x) \in \mathbb{R}\}$.

(i) Prove that $\mathcal{F}$ is a closed subspace of $(\mathbf{C}^0_b(\mathbb{R}), \|\cdot\|_\infty)$ and $\mathbf{C}^0_c(\mathbb{R}) \subset \mathbf{C}^0_0(\mathbb{R}) \subset \mathcal{F}$.

(ii) Let $L : \mathcal{F} \to \mathbb{R}$ be the linear functional defined as $L(f) := f(\infty)$ and prove that it can be extended to a functional $\tilde{L} \in (\mathbf{C}^0_b(\mathbb{R}), \|\cdot\|_\infty)'$ by means of the Hahn-Banach theorem.

(iii) Prove that there is no a finite Radon measure ν on $\mathbb{R}$, that is an element $\nu \in \mathcal{M}(\mathbb{R})$, such that $\tilde{L}(f) = \int_{\mathbb{R}} f\, d\nu$ for each $f \in \mathbf{C}^0_b(\mathbb{R})$. (**Hint:** Observe that $\tilde{L}(f) = 0$ for each $f \in \mathbf{C}^0_c(\mathbb{R})$).

We now consider the local weak*-convergence of Radon vector measures and we point out some relationships with the local weak*-convergence of their total variation. Before let us introduce an alternative approximation to Theorem 1.14 for positive Radon measures, which will need in the proof.

Lemma 1.117. *Let (X, d) be a l.c.s. metric space, let μ be a positive Radon measure on $(X, \mathcal{B}(X))$ and let $E \in \mathcal{B}_{\mathrm{comp}}(X)$ with $\mu(\partial E) = 0$. Then, for each $\epsilon > 0$ there exist a compact set K and an open set A (which may be empty) such that*

$$\bar{A} \subset E \subset \overset{\circ}{K} \text{ and } \mu(K \setminus A) < \epsilon\,.$$

Remark 1.118. Lemma 1.117 is a refinement of the approximation of Radon measures on a l.c.s metric space by means of compact sets from below and open sets from above (see Theorem 1.14).

Proof. Let us first observe that, since $\bar{E}$ is compact, by Lemma 1.17, there is a relatively compact open set V such that

$$\bar{E} \subset V \subset \bar{V}\,. \tag{1.91}$$

If h, m are integers and $\overset{\circ}{E} \neq \emptyset$ (otherwise choose $A = \emptyset$) , let us consider the sequences of sets

$$A_h := \left\{ x \in \overset{\circ}{E} : d(x, \partial E) > \frac{1}{h} \right\}, \quad K_m := \left\{ x \in \bar{V} : d(x, \bar{E}) \leqslant \frac{1}{m} \right\}.$$

Since $(A_h)_h$ is an increasing sequence of open sets, with $\bar{A}_h \subset \overset{\circ}{E}$ and $\overset{\circ}{E} = \cup_{h=1}^{\infty} A_h$, it follows that

$$\lim_{h\to\infty} \mu(A_h) = \mu(\overset{\circ}{E}) = \mu(E) < \infty\,.$$

Thus, if we take $A = A_h$ for h large enough, then, $\mu(E \setminus A) < \epsilon/2$. In the same way,

$$(K_m)_m \text{ is a decreasing sequence of compact sets, } \cap_{m=1}^{\infty} K_m = \bar{E}\,, \tag{1.92}$$

with

$$\overset{\circ}{K}_m \supset E\,. \tag{1.93}$$

Indeed it is immediate, by construction, that $(K_m)_m$ is a decreasing sequence of closed sets and, by (1.91), that $\cap_{m=1}^{\infty} K_m = \bar{E}$. Since each $K_m \subset \bar{V}$ and $\bar{V}$ is compact, K_m is compact, too and then (1.92). On the other hand, if $x_0 \in E$, then, by (1.91), $x_0 \in V$ and $d(x_0, \bar{E}) = 0 < 1/m$ for each m. Since V is open and the function $V \ni x \mapsto d(x, \bar{E})$ is continuous, it follows that $x_0 \in \overset{\circ}{K}_m$ for each m and then (1.93). Since $\mu(K_1) < \infty$, by (1.92),

$$\lim_{m\to\infty} \mu(K_m) = \mu(\bar{E}) = \mu(E)\,,$$

and, if we choose $K = K_m$ for m large enough, $\mu(K \setminus E) < \epsilon/2$ with $\overset{\circ}{K} \supset E$, by (1.93). □

Theorem 1.119. *Let $(\nu_h)_h$ and ν be $\mathbb{R}^m$-valued Radon vector measures, that is ν_h, $\nu : \mathcal{B}(X) \to \mathbb{R}^m$, and let μ a positive Radon measure on a l.c.s. metric space (X, d).*

(i) *If $\nu_h \overset{*}{\rightharpoonup} \nu$, then for every open set $A \subset X$*

$$|\nu|(A) \leqslant \liminf_{h\to\infty} |\nu_h|(A)\,. \tag{1.94}$$

(ii) *If $\nu_h \overset{*}{\rightharpoonup} \nu$ and $|\nu_h| \overset{*}{\rightharpoonup} \mu$, then*

$$|\nu|(B) \leqslant \mu(B) \quad \forall\, E \in \mathcal{B}(X)\,. \tag{1.95}$$

Moreover, if $E \in \mathcal{B}_{\text{comp}}(X)$ with $\mu(\partial E) = 0$, then

$$\nu(E) = \lim_{h\to\infty} \nu_h(E)\,.$$

(iii) *If $\nu_h \overset{*}{\rightharpoonup} \nu$ and $\lim_{h\to\infty} |\nu_h|(X) = |\nu|(X) < \infty$, then* (NC) *holds with $\mu_h = |\nu_h|$ and $\mu = |\nu|$. In particular $|\nu_h| \overset{*}{\rightharpoonup} |\nu|$.*

Proof. (i): Let us recall that, by Riesz representation theorem 1.73, $\nu_h = (\nu_h^1, \ldots, \nu_h^m) = w_{\nu_h} |\nu_h|$, $\nu = (\nu_1, \ldots, \nu_m) = w_\nu |\nu|$ with $w_{\nu_h}, w_\nu : X \to \mathbb{R}^m$ Borel measurable, $|w_{\nu_h}| = |w_\nu| = 1$, $|\nu_h|$-a.e. and $|\nu|$-a.e. in X, respectively, and, for each open set $A \subset X$,

$$|\nu|(A) = \sup\left\{ \int_X (w_u, \psi)_{\mathbb{R}^m}\, d|\nu| : \psi \in (\mathbf{C}_c^0(A))^m,\ \|\psi\|_\infty \leqslant 1 \right\}. \tag{1.96}$$

By the assumptions, for each $\varphi \in \mathbf{C}_c^0(X)$

$$\int_X \varphi\, w_\nu\, d|\nu| = \int_X \varphi\, d\nu = \lim_{h\to\infty} \int_X \varphi\, d\nu_h = \lim_{h\to\infty} \int_X \varphi\, w_{\nu_h}\, d|\nu_h| \quad \forall \varphi \in \mathbf{C}_c^0(X)\,.$$

This implies that for each $\psi \in (\mathbf{C}_c^0(A))^m$ with $\|\psi\|_\infty \leqslant 1$

$$\int_X (w_\nu, \psi)_{\mathbb{R}^m}\, d|\nu| = \lim_{h\to\infty} \int_X (w_{\nu_h}, \psi)_{\mathbb{R}^m}\, d|\nu_h| \leqslant \liminf_{h\to\infty} |\nu_h|(A)$$

Since $\psi \in (\mathbf{C}_c^0(A))^m$ with $\|\psi\|_\infty \leqslant 1$ is arbitrary, by (1.96), (1.94) follows.

(ii): Let $A \subset X$ be a relatively compact open set and, $A_t := \{x \in A :\, d(x, \partial A) > t\}$ and let $u \in \mathbf{C}_c^0(A)$ such that $\chi_{A_t} \leqslant u \leqslant \chi_A$. Then, by (1.94), we have

$$|\nu|(A_t) \leqslant \liminf_{h\to\infty} |\nu_h|(A_t) \leqslant \liminf_{h\to\infty} \int_X u\, d|\nu_h| = \int_X u\, d\mu \leqslant \mu(A)\,.$$

By letting $t \to 0^+$, we get $|\nu|(A) \leqslant \mu(A)$, and since A is arbitrary, inequality (1.95) follows from Theorem 1.14. Let us now prove that $\nu(E) = \lim_{h\to\infty} \nu_h(E)$ whenever $E \in \mathcal{B}_{\mathrm{comp}}(X)$ with $\mu(\partial B) = 0$. Given $\epsilon > 0$, by Lemma 1.117, we find an open set A and a compact set K such that $\bar{A} \subset E \subset \overset{\circ}{K}$ and $\mu(K \setminus A) \leqslant \epsilon$. Then, for every $u \in \mathbf{C}_c^0(\overset{\circ}{K})$, $0 \leqslant u \leqslant 1$ with $u = 1$ on A, we find

$$\left| \int_X u\, d\nu_h - \nu_h(E) \right| \leqslant \int_X |u - \chi_E|\, d|\nu_h| \leqslant |\nu_h|(K \setminus A)\,,$$

$$\left| \int_X u\, d\nu - \nu(E) \right| \leqslant |\nu|(K \setminus A) \leqslant \mu(K \setminus A)\,,$$

$$\lim_{h\to\infty} \left| \int_X u\, d\nu_h - \int_X u\, d\nu \right| = 0\,.$$

Since $|\nu_h| \overset{*}{\rightharpoonup} \mu$ and $K \setminus A$ is compact, by (1.83), we have $\limsup_{h\to\infty} |\nu_h|(K \setminus A) \leqslant |\nu|(K \setminus A) \leqslant \mu(K \setminus A)$. Recalling $\mu(K \setminus A) \leqslant \epsilon$, we thus conclude that

$$\begin{aligned}\limsup_{h\to\infty} |\nu_h(E) - \nu(E)| \leqslant \limsup_{h\to\infty} \left| \nu_h(E) - \int_X u\, d\nu_h \right| + \limsup_{h\to\infty} \left| \int_X u\, d\nu_h - \int_X u\, d\nu \right| \\ + \limsup_{h\to\infty} \left| \int_X u\, d\nu - \nu(E) \right| \leqslant 2\epsilon, \quad \forall \epsilon > 0\,.\end{aligned}$$

(iii): Without loss of generality, we can assume that both $|\nu_h|(X) < \infty$ for each h and $|\nu|(X) < \infty$, that is that $(\nu_h)_h$ and ν are measures contained in $(\mathcal{M}(X))^m$. Thus, by claim (i), it follows that the assumptions of Proposition 1.115 are satisfied and then the proof is accomplished.

□

Remark 1.120. A typical application of statement (ii) of Theorem 1.119 (or also of statement (iii) of Theorem 1.113) is the following: let us consider an increasing family $(A_t)_t$ of relatively compact open sets labelled on an interval I such that $\bar{A}_s \subset A_t$, for $s < t$. Then,

Exercise: $\mu(\partial A_t) = 0$ except for countably many $t \in I$.
(*Hint:* Let $(V_i)_i$ be an increasing sequence of relaatively compact open sets such that $X = \cup_{i=1}^\infty V_i$ (see Lemma 1.17). Since ∂A_t with $t \in I$ are pairwise disjoint, by the additivity of μ, prove that , for given $\epsilon > 0$ and $i \in \mathbb{N}$ the set

$$\{t \in I :\, \mu(\partial A_t) > \epsilon,\; A_t \subset V_i\}\,.$$

Then deduce the desired conclusion.)

Hence, by Theorem 1.119 (ii) $\nu_h(A_t) \to \nu(A_t)$ for $\mathcal{L}^1$-a.e. $t \in I$.

The classical De La Vallée Poussin compactness criterion for finite Radon measures easily follows by the sequential weak*-compactness of bounded sets in a dual space of a separable normed vector space (see, for instance, [SC, Theorem 3.30]) and the characterization of the space of finite Radon measures (Theorem 1.83).

Theorem 1.121 (Weak*-compactness)**.** *If* $(\nu_h)_h$ *is a sequence of* $\mathbb{R}^m$*- valued finite Radon measures on the l.c.s. metric space* X*, that is* $(\nu_h)_h \subset (\mathcal{M}(X))^m$*, with* $\sup_h |\nu_h|(X) < \infty$*, then it has a weakly*- converging subsequence. Moreover, the map* $\nu \mapsto |\nu|$ *is lower semicontinuous with respect to the weak*-convergence, that is, assume that* $(\nu_h)_h$ *weakly*-converges to* ν*, then*

$$|\nu|(X) \leqslant \liminf_{h\to\infty} |\nu_h|(X)\,.$$

.

The previous theorem can be used to get immediately a corresponding result in the frame of local weak*-convergence.

Corollary 1.122 (Local weak* compactness)**.** *Let* $(\nu_h)_h$ *be a sequence of* $\mathbb{R}^m$*- valued Radon measures on the l.c.s. metric space* X*,* $(\nu_h)_h \subset (\mathcal{M}_{loc}(X))^m$*, such that*

$$\sup\{|\nu_h|(K) : h \in \mathbb{N}\} < \infty$$

for every compact $K \subset X$*; then it has a locally weakly*-converging subsequence.*

Proof. Let $(V_i)_i$ be a sequence of relatively compact open sets such that

$$X = \cup_{i=1}^\infty V_i \text{ and } V_i \subset V_{i+1} \text{ for each } i \in \mathbb{N}\,.$$

For given i, let $(\nu_h^i)_h$ be the sequence of $\mathbb{R}^m$-valued **finite** Radon measures defined as

$$\nu_h^i := \nu_h \llcorner \bar{V}_i\,.$$

Since, by our assumption,

$$\sup_h |\nu_h^i|(X) = \sup_h |\nu_h|(V_i) < \infty\,.$$

By Theorem 1.121 and by means of a standard diagonal process, there exist a subsequence $(h_k)_k$ and finite Radon measure $\nu^i \in (\mathcal{M}(X))^m$ such that $(\nu_{h_k} \llcorner V_i)_k$ weakly* converges to ν^i, that is

$$\int_{V_i} \varphi\, d\nu_{h_k} = \int_X \varphi\, d(\nu_{h_k} \llcorner V_i) \to \int_X \varphi\, d\nu^i \quad \text{for each } \varphi \in \mathbf{C}_0^0(X), i \in \mathbb{N}\,. \tag{1.97}$$

Moreover, by definition of $\mathbb{R}^m$-valued finite Radon measure (see Definition 1.74 (ii)), thete exists a Borel measurable functions $w_i : X \to \mathbf{S}^{m-1}$ such that

$$\nu^i = w_i |\nu^i| \text{ with } |\nu^i| \text{ positive finite Radon measure on } X, \text{ for each } i \in \mathbb{N}\,.$$

Let us now prove that, for each $i \in \mathbb{N}$,

$$\nu^i \llcorner V_i = \nu^{i+1} \llcorner V_i\,,$$

that is, there exists a finte positive Radon measure on X such that

$$|\nu^i| \llcorner V_i = |\nu^{i+1}| \llcorner V_i = \mu \text{ and } w_i(x) = w_{i+1}(x)\ \mu\text{-a.e. } x \in V_i\,. \tag{1.98}$$

Fix $i \in \mathbb{N}$ and let us consider a function $\varphi \in \mathbf{C}^0_c(V_i)$. Then we can infer that

$$\int_X \varphi \, d\nu_{h_k} \llcorner V_i = \int_{V_i} \varphi \, d\nu_{h_k} = \int_{V_{i+1}} \varphi \, d\nu_{h_k} = \int_X \varphi \, d(\nu_{h_k} \llcorner V_{i+1}) \, .$$

Passing to the limit as $k \to \infty$ in the previous identity, by (1.97), we get

$$\begin{aligned} \int_X \varphi \, w_i \, d|\nu^i| &= \int_X \varphi \, d\nu^i = \int_X \varphi \, d\nu^{i+1} \\ &= \int_X \varphi \, w_{i+1} \, d|\nu^{i+1}| \text{ for each } \varphi \in \mathbf{C}^0_c(V_i), \; i \in \mathbb{N} \, . \end{aligned}$$

Let $w_i = (w_{i,1}, \ldots, w_{i,m})$, then we can rewrite the previous identity as

$$\int_{V_i} \varphi \, w_{i,j} \, d|\nu^i| = \int_X \varphi \, w_{i,j} \, d|\nu^i| = \int_X \varphi \, w_{i+1,j} \, d|\nu^{i+1}| = \int_{V_i} \varphi \, w_{i+1,j} \, d|\nu^{i+1}| \tag{1.99}$$

for each $\varphi \in \mathbf{C}^0_c(V_i)$, $i \in \mathbb{N}$, $j = 1, \ldots, m$, which implies that, for each $u = (u_1, \ldots, u_m) \in (\mathbf{C}^0_c(V_i))^m$ and for each $i \in \mathbb{N}$,

$$\int_X (w_i, u)_{\mathbb{R}^m} \, d|\nu^i| = \int_X (w_{i+1}, u)_{\mathbb{R}^m} \, d|\nu^{i+1}| \tag{1.100}$$

By (1.100) and the representation of the total variation of a $\mathbb{R}^m$-valued Radon measure (see (1.33))we get that

$$|\nu^i|(A) = |\nu^{i+1}|(A) \text{ for each open set } A \subset V_i \, . \tag{1.101}$$

By the approxomiation with open sets for a positive Radon measure, we can infer that

$$|\nu^i| \llcorner V_i = |\nu^{i+1}| \llcorner V_i = \mu \, . \tag{1.102}$$

Therefore, by (1.102), we can rewrite (1.99) as follows

$$\int_{V_i} \varphi \, w_{i,j} \, d\mu = \int_{V_i} \varphi \, w_{i+1,j} \, d\mu \text{ for each } \varphi \in \mathbf{C}^0_c(V_i), \; i \in \mathbb{N}, \; j = 1, \ldots, m \, . \tag{1.103}$$

By Remark 1.64 , for given $i \in \mathbb{N}$, for each Borel set $E \subset V_i$ there exists a sequence $(\varphi_h)_h \subset \mathbf{C}^0_c(V_i)$, such that

$$\varphi_h \to \chi_E \text{ in } L^1(V_i, \mu) \text{ and } |\varphi_h(x)| \leqslant 1 \text{ for each } x \in V_i \, . \tag{1.104}$$

By (1.103), (1.104) and the Lebesgue dominated convergence theorem, we can infer that

$$\int_E w_{i,j} \, d\mu = \int_E w_{i+1,j} \, d\mu \text{ for each Borel set } E \subset V_i, \; i \in \mathbb{N}, \; j = 1, \ldots, m \, ,.$$

which is equivalent to

$$w_i(x) = w_{i+1}(x) \quad \mu\text{-a.e. } x \in V_i \, . \tag{1.105}$$

By (1.102) and (1.105), (1.98) follows. Let us now define a positive Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ and a Borel measurable function $w : X \to \mathbf{S}^{m-1}$ as follows:

$$\mu(E) := |\nu^i|(E) \text{ if } E \in \mathcal{B}(V_i) \text{ for some } i$$

and

$$w(x) := w_i(x) \text{ if } x \in V_i \text{ for some } i \, .$$

By (1.98) both μ and w are well defined. Let us define the $\mathbb{R}^m$-valued Radon measure $\nu : \mathcal{B}_{\text{comp}}(X) \to \mathbb{R}^m$

$$\nu(E) = \int_E w \, d\mu \text{ if } E \in \mathcal{B}_{\text{comp}}(X) .$$

Then by (1.97), it follows that $(\nu_{h_k})_k$ locally weak* converges to ν and we accomplish the proof. □

Regularization of Radon measures in $\mathbb{R}^n$.

In this section we assume that $X = \mathbb{R}^n$, endowed with the Euclidean metric, and we are going to deal with the approximation of a given $\mathbb{R}^m$-valued Radon vector measure ν on $\mathbb{R}^n$ by means of a sequence of $\mathbb{R}^m$-valued Radon vector measures

$$\nu_h = \mathcal{L}^n_{w_h} \quad \text{on } \mathbb{R}^n$$

(see (1.19)) with $w_h \in \mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $\nu_h \overset{*}{\rightharpoonup} \nu$.

We saw that, given $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$, there exists $(f_h)_h \subset \mathbf{C}^0_c(\mathbb{R}^n)$ such that $f_h \to f$ in $L^p(\mathbb{R}^n)$ (see Theorem 1.63). Since the differential structure of $\mathbb{R}^n$, we are going to improve this approximation, looking for an approximation by regular $\mathbf{C}^\infty$-functions on $\mathbb{R}^n$. A powerful tool for getting such a goal is the so-called *approximation by convolution*, which we will briefly recall here.

Let us first recall the notion of *support for a (Lebesgue) measurable function.*

Let us recall (see (1.6)) that given a function $f : \mathbb{R}^n \to \mathbb{R}$, its support is the set

$$\text{(S)} \qquad \text{spt}(f) := \overline{\{x \in \Omega : \, f(x) \neq 0\}} .$$

This definition is not suitable for a (Lebesgue) measurable function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Indeed we would like that this notion satisfies the following property:

$$f_1 = f_2 \quad \text{a.e. in } \mathbb{R}^n \quad \Rightarrow \quad \text{spt}(f_1) = \text{spt}(f_2) , \quad \text{except for a negligible set.}$$

But this is not the case. Indeed

Example: Let $f_1 := \chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ and $f_2 \equiv 0$. Then it is clear that

$$f_1 = f_2 \quad \text{a.e. in } \mathbb{R}$$

but

$$\text{spt}(f_1) = \overline{\mathbb{Q}} = \mathbb{R} \quad \text{and} \quad \text{spt}(f_2) = \emptyset .$$

Proposition 1.123 (Essential support of a function)**.** *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Denote*

$$\mathcal{A}_f := \{\omega \subset \mathbb{R}^n : \omega \text{ open set and } f = 0 \text{ a.e. in } \omega\}$$

and let

$$A_f := \cup_{\omega \in \mathcal{A}_f} \omega .$$

Then A_f is an open set and

$$f = 0 \ \text{ a.e. in } A_f .$$

The closed set

$$\text{(ES)} \qquad \text{spt}_e(f) := \mathbb{R}^n \setminus A_f$$

is called the essential support of f in $\mathbb{R}^n$.

Proof of Proposition 1.123. See [B, Proposition 4.17] □

Remark 1.124. (i) From definition (ES), it follows that, if $f_1 = f_2$ a.e. in $\mathbb{R}^n$, then $\mathrm{spt}_e(f_1) = \mathrm{spt}_e(f_2)$.
(ii) Definitions (S) and (ES) agree when the function is continuous. More precisely **Exercise:** If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then

$$\mathbb{R}^n \setminus A_f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}\,.$$

Definition 1.125 (Friedrichs' mollifiers, 1944)**.** *A sequence of* <u>*mollifiers*</u> *is a sequence of functions $\varrho_h : \mathbb{R}^n \to \mathbb{R}$ ($h = 1, 2, \dots$) such that, for each h,*

(Mo1) $$\varrho_h \in \mathbf{C}^\infty(\mathbb{R}^n)\,;$$

(Mo2) $$\mathrm{spt}(\varrho_h) \subset B(1/h)\,;$$

(Mo3) $$\int_{\mathbb{R}^n} \varrho_h\, dx = 1\,;$$

(Mo4) $$\varrho_h(x) \geqslant 0 \quad \forall\, x \in \mathbb{R}^n\,.$$

Example of mollifiers: It is quite simple constructing a sequence of mollifiers, starting from a given <u>non vanishing</u> function $\varrho : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\varrho \in \mathbf{C}_c^\infty(\mathbb{R}^n)\,,\ \mathrm{spt}(\varrho) \subset B(1),\ \ \varrho \geqslant 0\,.$$

For instance, let

$$\varrho(x) := \begin{cases} \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1 \end{cases}\,.$$

Then it is easy to see that $\varrho \in \mathbf{C}_c^\infty(\mathbb{R}^n)$. Moreover we yield a sequence of mollifiers by defining

(1.106) $$\varrho_h(x) := c\, h^n\, \varrho(h\, x) \quad x \in \mathbb{R}^n,\ h \in \mathbb{N}$$

and

$$c := \left(\int_{\mathbb{R}^n} \varrho\, dx\right)^{-1}\,.$$

Remark 1.126. Observe that, without loss of generality, by (1.106), we can assume that a sequence of mollifiers $(\varrho_h)_h$ satisfies the simmetry condition

(1.107) $$\varrho_h(-x) = \varrho_h(x) \quad \forall\, x \in \mathbb{R}^n,\ h \in \mathbb{N}$$

Notation: If A, $B \subset \mathbb{R}^n$, $A \pm B$ denotes the set

$$A \pm B := \{a \pm b : \ a \in A,\ b \in B\,\}$$

Exercise: Prove that
(i) if A is compact and B is closed, then $A + B$ is closed;
(ii) if A and B are compact so is $A + B$

Proposition 1.127 (Definition and first mollifiers' properties)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and let $(\varrho_h)_h$ be a sequence of mollifiers. Define, for given $h \in \mathbb{N}$ and $x \in \mathbb{R}^n$,*

$$f_h(x) = (\varrho_h * f)(x) := \int_{\mathbb{R}^n} \varrho_h(x-y)\, f(y)\, dy \quad x \in \mathbb{R}^n.$$

Then

(i) *the function $f_h : \mathbb{R}^n \to \mathbb{R}$ is well defined;*
(ii) *$f_h(x) = (\varrho_h * f)(x) = (f * \varrho_h)(x)$ for all $x \in \mathbb{R}^n$ and $h \in \mathbb{N}$;*
(iii) *$f_h \in \mathbf{C}^0(\mathbb{R}^n)$ for each h;*
(iv) *if $f \in \mathbf{C}^0(\mathbb{R}^n)$, then $\varrho_h * f \to f$ uniformly on compact sets of $\mathbb{R}^n$, as $h \to \infty$.*

The function f_h is called h^{th}- mollifier of f.

Proof. See [B, Propositions 4.19 and 4,21] and [SC, Proposition 2.68 and Lemma 2.74] □

Remark 1.128. The symbol $*$ denotes the convolution product between two functions defined on the whole $\mathbb{R}^n$. Notice also that the conclusions of Proposition 1.127 still holds if $f \in L^1_{loc}(\mathbb{R}^n)$ and $\varrho \equiv \varrho_h \in \mathbf{C}^0(\mathbb{R}^n)$ satisfying (Mo2). Actually, it is possible to define the convolution product between two functions $g \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$ and $f \in L^1(\mathbb{R}^n)$

$$(g * f)(x) := \int_{\mathbb{R}^n} g(x-y)\, f(y)\, dy$$

and it holds that (see [GZ, Theorem 6.51])

$$(g * f) \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|g * f\|_{L^p(\mathbb{R}^n)} \leqslant \|g\|_{L^p(\mathbb{R}^n)}\, \|f\|_{L^1(\mathbb{R}^n)}\,.$$

Theorem 1.129 (Friedrichs-Sobolev, approximation by convolution in L^p)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(\varrho_h)_h$ be a sequence of mollifiers. Then*

(i) *$f * \varrho_h \in C^\infty(\mathbb{R}^n)$ for each $h \in \mathbb{N}$.*
(ii) *$\|f * \varrho_h\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}$ for each $h \in \mathbb{N}$, $f \in L^p(\mathbb{R}^n)$, for every $p \in [1, \infty]$.*
(iii) *$\mathrm{spt}(f * \varrho_h) \subset \mathrm{spt}_e(f) + B(1/h)$ for each $h \in \mathbb{N}$.*
(iv) *If $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, then $f * \varrho_h \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for each $h \in \mathbb{N}$, and $f * \varrho_h \to f$ as $h \to \infty$, in $L^p(\mathbb{R}^n)$, provided that $1 \leqslant p < \infty$.*

Proof of Theorem 1.129. See [B, Proposition 4.20 and Theorem 4.22] and [SC, Theorem 2.70]. □

Historical notes: Mollifiers were introduced by K. Friedrichs in 1944, which are, according to P. Lax, a watershed in the modern theory of PDEs. However, S. Sobolev had used mollifiers in his epoch making 1938 paper [So] (the paper containing the proof of the Sobolev embedding theorem), as Friedrichs himself acknowledged in later papers.

Let us now come back to the approximation of a Radon vector measure. Let ν be a $\mathbb{R}^m$-valued Radon vector measure and $(\varrho_h)_h$ be a sequence of mollifiers. Let us then define the sequence of functions $\nu * \varrho_h : \mathbb{R}^n \to \mathbb{R}^m$ $(h \in \mathbb{N})$ as

$$(\nu * \varrho_h)(x) := \int_{\mathbb{R}^n} \varrho_h(x-y)\, d\nu(y) \quad \text{if } x \in \mathbb{R}^n\,. \tag{1.108}$$

Theorem 1.130 (Approximation of Radon vector measures)**.** *Let $\nu = (\nu_1, \ldots, \nu_m)$ be a Radon vector measure in $\mathbb{R}^n$ and let $(\varrho_h)_h$ be a sequence of mollifiers which also satisfies simmetry condition* (1.106)*. Then*

(i) *The functions $\nu * \varrho_h : \mathbb{R}^n \to \mathbb{R}^m$ belong to $(\mathbf{C}^\infty(\mathbb{R}^n))^m$ and $\nabla^\alpha(\nu * \varrho_h) = \nu * \nabla^\alpha \varrho_h$ for any $\alpha \in \mathbb{N}^n$.*

(ii) *If $\nu_h := \mathcal{L}^n_{\nu * \varrho_h}$, the sequence of measures $(\nu_h)_h$ locally weakly*converges in $\mathbb{R}^n$ to ν as $h \to \infty$ and, for each $E \in \mathcal{B}(\mathbb{R}^n)$, it holds the estimate*

$$|\nu_h|(E) = \int_E |\nu * \varrho_h| \, dx \leqslant |\nu|(I_{1/h}(E)) \, .$$

(iii) *The sequence of measures $(|\nu_h|)_h$ locally weakly*converges in $\mathbb{R}^n$ to $|\nu|$ as $h \to \infty$.*

Proof. (i): The first statement can be easily proved as in the proof of Theorem 1.129 (i), by induction on the length of α by using a difference quotient argument and passing to the limit under the integral.

(ii): Let us first note that, using Fubini's theorem and simmetry condition (1.106), it is easily seen that

$$\int_{\mathbb{R}^n} (\nu * \varrho_h) \, v \, dx = \int_{\mathbb{R}^n} (v * \varrho_h) \, d\nu \tag{1.109}$$

for each $v \in L^1(\mathbb{R}^n)$. Thus, if $u \in \mathbf{C}^0_c(\mathbb{R}^n)$, by (1.109) and Proposition 1.127 (iv),

$$\int_{\mathbb{R}^n} u \, d\nu_h = \int_{\mathbb{R}^n} u \, (\nu * \varrho_h) \, dx = \int_{\mathbb{R}^n} (u * \varrho_h) \, d\nu \to \int_{\mathbb{R}^n} u \, d\nu \text{ as } h \to \infty \, .$$

Let $E \in \mathcal{B}(\mathbb{R}^n)$ and let us estimate $|\nu_h|$: by (1.20) and for Fubini's theorem

$$\begin{aligned} |\nu_h|(E) &= \int_E |\nu * \varrho_h| \, dx = \int_E \left| \int_{\mathbb{R}^n} \varrho_h(x-y) \, d\nu(y) \right| \\ &\leqslant \int_E \left(\int_{\mathbb{R}^n} \varrho_h(x-y) \, d|\nu|(y) \right) dx = \int_{\mathbb{R}^n} \left(\int_E \varrho_h(x-y) \, dx \right) d|\nu|(y) \\ &\leqslant \int_{I_{1/h}(E)} \left(\int_E \varrho_h(x-y) \, dx \right) d|\nu|(y) \leqslant |\nu|(I_{1/h}(E)) \, . \end{aligned}$$

(iii): Let $A_t := U(t)$ if $t \in I := (0, \infty)$. By Remark 1.120, we can find an increasing sequence of open sets $A_k \Subset \mathbb{R}^n$ such that $\mathbb{R}^n = \cup_{k=1}^\infty A_k$ and $|\nu|(\partial A_k) = 0$ for each $k \in \mathbb{N}$. As a consequence of (ii),

$$\limsup_{h \to \infty} |\nu_h|(A_k) \leqslant |\nu|(I_0(A_k)) = |\nu|(\bar{A}_k) = |\nu|(A_k) \quad \text{ for each } k \, .$$

On the other hand, Theorem 1.119 (i) implies that $\liminf_{h \to \infty} |\nu_h|(A) \geqslant |\nu|(A)$ for any open set $A \subset \mathbb{R}^n$. By Proposition 1.115, we infer that $(|\nu_h|)_h$ weakly* converges to $|\nu|$ in A_k, and since k is arbitrary the statement follows.

□

2. Differentiation of Radon measures ([AFP, Ma])

Motivation: In this section we are going to introduce the main results about the differentiation of measures which will be used later in the rectifiability and in the study of sets of finite perimeter. One of the main goals is to prove the following result.

Theorem 2.1 (of Lebesgue points)**.** *: Let μ be a positive Radon measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and let $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$. Then μ-a.e. $x \in \text{spt}(\mu)$ there exists*

$$\text{(LP)} \qquad \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\mu(y) = 0\,.$$

Definition 2.2. A point $x \in \mathbb{R}^n$ for which (LP) holds is called a Lebesgue point of f.

2.1. Covering theorems and Vitali-type covering property for measures on $\mathbb{R}^n$.

We begin to introduce two types of covering theorems in $X = \mathbb{R}^n$. The difference between them is that the first ones (Vitali's coverings) apply to a larger class of coverings and a narrower class of measures whereas in the second type the coverings (Besicovitch's coverings) are more restricted but the measures can be very general; for example all Radon measures on $\mathbb{R}^n$ are included. In both cases we first prove a geometric result on collections of balls in $\mathbb{R}^n$ and then apply it to get a Vitali-type covering theorem for measures.

Let us begin with some notions on coverings in a general metric space.

To begin with, let us agree that by *disjoint family of subsets of a metric space* (X, d) we mean a family $\mathcal{F}$ such that $E \cap F = \emptyset$ whenever E, $F \in \mathcal{F}$ and $E \neq F$; we set also

$$\cup\mathcal{F} := \cup_{E\in\mathcal{F}} E\,.$$

Definition 2.3. Let (X, d) be a metric space.

(i) A family $\mathcal{F}$ of closed balls of (X, d) is a *cover of a set* $A \subset X$ if

$$A \subseteq \cup\mathcal{F}\,.$$

(ii) A family $\mathcal{F}$ of closed balls of (X, d) is a *fine cover of* A, or also that $\mathcal{F}$ covers A *in the sense of Vitali*, if it is a cover of A and, for each $x \in A$,

$$\text{(2.1)} \qquad \inf\{d(B) :\ B \in \mathcal{F},\ x \in B\} = 0\,.$$

Vitali covering theorem and the Lebesgue measure.

Theorem 2.4 (Vitali covering theorem)**.** *Let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$ with*

$$D = \sup\{d(B) :\ B \in \mathcal{G}\} < \infty\,.$$

Then there exists a (pairwise) disjoint family $\mathcal{F} \subseteq \mathcal{G}$, which is at most countable, such that

$$\cup_{B\in\mathcal{G}} B \subset \cup_{B\in\mathcal{F}} \hat{B}\,.$$

where $\hat{B}$ is an enlargement of B, that is $\hat{B} = 5B$.

Before the proof of Vitali's covering theorem, let us recall the Hausdorff Maximal Principle (see [GZ, Theorem 1.4]).

Theorem 2.5 (Hausdorff Maximal Principle)**.** *If $\mathcal{S}$ is a family of sets (or a collection of families of sets) and if $\cup\{E : E \in \mathcal{E}\} \in \mathcal{S}$ for any subfamily $\mathcal{E}$ of $\mathcal{S}$ totally ordered with respect to the inclusion, that is with the property that*

$$E_1 \subset E_2 \text{ or } E_2 \subset E_1 \text{ whenever } E_1,\, E_2 \in \mathcal{E}\,.$$

Then there exists $E^* \in \mathcal{S}$, which is maximal in the sense that it is not a subset of any other member of $\mathcal{S}$.

Proof of Theorem 2.4. Let us define the sequence of subfamilies of $\mathcal{G}$

$$\mathcal{G}_j := \left\{ B \in \mathcal{G} : \frac{D}{2^j} < d(B) \leqslant \frac{D}{2^{j-1}} \right\} \quad j = 1, 2, \ldots$$

Then it is trivial to see that

$$\mathcal{G} = \cup_{j=1}^{\infty} \mathcal{G}_j\,, \tag{2.2}$$

$$\mathcal{G}_j \cap \mathcal{G}_{j'} = \emptyset \quad \forall j \neq j'\,. \tag{2.3}$$

Let us define inductively a subfamily $\mathcal{F}_j \subset \mathcal{G}_j$ as follows.

Let $\mathcal{F}_1 \subset \mathcal{G}_1$ be a maximal subcolletion of pairwise disjoint elements if $\mathcal{G}_1 \neq \emptyset$. For if not, let $\mathcal{F}_1 = \emptyset$.

Exercise: Prove that such a family $\mathcal{F}_1$ exists by means of the Hausdorff Maximal Principle.

Assuming that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{j-1}$ have been chosen, let $\mathcal{F}_j$ be a maximal pairwise disjoint family of the family

$$\mathcal{G}_j^* := \left\{ B \in \mathcal{G}_j : B \cap B' = \emptyset, \forall B' \in \cup_{i=1}^{j-1} \mathcal{F}_i \right\}$$

if $\mathcal{G}_j^* \neq \emptyset$; for if not let $\mathcal{F}_j := \emptyset$. Let

$$\mathcal{F} = \cup_{j=1}^{\infty} \mathcal{F}_j\,.$$

From (2.2) and (2.3), $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{F}$ is a disjoint collection of balls. Moreover $\mathcal{F}$ is at most countable, since the pairwise disjoint family of open balls $\{\overset{\circ}{B} : B \in \mathcal{F}\}$ has to be at most countable (why?).

Let us observe that,

$$\text{for fixed } B \in \mathcal{G}_j \; \exists \; B_1 \in \cup_{i=1}^{j} \mathcal{F}_i \text{ such that } B_1 \cap B \neq \emptyset\,. \tag{2.4}$$

Indeed, for if not, the family

$$\mathcal{F}_j^* = \mathcal{F}_j \cup \{B_1\}$$

would be a disjoint family of $\mathcal{G}_j^*$, thus contradicting the maximality of $\mathcal{F}_j$. Moreover

$$d(B) \leqslant \frac{D}{2^{j-1}} = 2\frac{D}{2^j} < 2\,d(B_1)\,, \tag{2.5}$$

which implies that

$$B \subset \hat{B}_1\,. \tag{2.6}$$

Indeed, if $B = B(x, r)$, $B_1 = B(x_1, r_1)$, let $z \in B$ and, since (2.4), there exists $y \in B \cap B_1$. Thus, by (2.5),

$$|z - x_1| \leqslant |z - y| + |y - x_1| \leqslant d(B) + r_1 \leqslant 2\,d(B_1) + r_1 \leqslant 4\,r_1 + r_1 = 5r_1\,.$$

and then (2.6) follows. □

Remark 2.6. (i) We emphasize that the point of the lemma is that the subcollection consists of countable disjoint elements, a very important consideration since countable additivity plays a central role in measure theory.

(ii) If $D = \sup\{d(B) : B \in \mathcal{G}\} = \infty$, Vitali's covering may fail. For instance **Exercise:** Let $\mathcal{G} := \{B(h) : h \in \mathbb{N}\}$, then prove that there is no a pairwise disjoint subfamily $\mathcal{F} \subset \mathcal{G}$ such that $\cup_{B\in\mathcal{G}} B \subset \cup_{B\in\mathcal{F}} \hat{B}$.

(iii) If the cover $\mathcal{G}$ is composed of open balls, the conclusion of Vitali's covering theorem still holds.

Now we are going to show, even though $\mathcal{F}$ could not cover the whole A, at least, it covers *almost all of* A with respect to the Lebesgue measure, that is, the so- called *Vitali covering property holds for the Lebesgue measure.*

Theorem 2.7 (Vitali covering property for the Lebesgue measure)**.** *Let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$, which is a fine cover of a (possibly non measurable) set $A \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Then there exists a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that*

$$\mathcal{L}^n\left(A \setminus \cup\mathcal{F}\right) = 0\,,$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue outer measure.

Proof. **1st step:** Suppose A is bounded with $0 < \mathcal{L}^n(A) < \infty$, otherwise we are done. Since $\mathcal{L}^n$ is a Borel regular outer measure, by Corollary 1.11, there is an open set $U_0 \subset \mathbb{R}^n$ such that $U_0 \supset A$ and

$$\mathcal{L}^n(U_0) \leqslant (1 + 7^{-n})\,\mathcal{L}^n(A)\,. \tag{2.7}$$

Let

$$\mathcal{G}_0 := \{B \in \mathcal{G} : B \subset U_0,\, d(B) \leqslant 1\}\,.$$

Being $\mathcal{G}$ fine, $\mathcal{G}_0$ is still a fine cover of A. Thus, by Vitali's covering theorem 2.4, there exists a disjoint, at most countable, subfamily $\mathcal{F}_0 \subset \mathcal{G}_0 \subset \mathcal{G}$ such that

$$A \subset \cup_{B\in\mathcal{G}} B \subset \cup_{B\in\mathcal{F}_0} \hat{B}\,.$$

Then

$$6^{-n}\,\mathcal{L}^n(A) < 5^{-n}\,\mathcal{L}^n(A) \leqslant 5^{-n} \sum_{B\in\mathcal{F}_0} \mathcal{L}^n(5B) = \sum_{B\in\mathcal{F}_0} \mathcal{L}^n(B)\,. \tag{2.8}$$

From (2.8), there exists a finite family of balls $\mathcal{F}_1 := \{B_1, \dots, B_{k_1}\} \subset \mathcal{F}_0 \subset \mathcal{G}_0$ such that

$$6^{-n}\,\mathcal{L}^n(A) \leqslant \sum_{i=1}^{k_1} \mathcal{L}^n(B_i)\,. \tag{2.9}$$

Define

$$A_1 := A \setminus \cup\mathcal{F}_1 = A \setminus \cup_{i=1}^{k_1} B_i\,.$$

If $\mathcal{L}^n(A_1) = 0$, we are done. Otherwise, from (2.7) and (2.9), we have that

$$\begin{aligned} \mathcal{L}^n(A_1) &\leqslant \mathcal{L}^n(U_0 \setminus \cup_{i=1}^{k_1} B_i) = \mathcal{L}^n(U_0) - \sum_{i=1}^{k_1} \mathcal{L}^n(B_i) \\ &\leqslant (1 + 7^{-n} - 6^{-n})\,\mathcal{L}^n(A) = u\,\mathcal{L}^n(A)\,, \end{aligned} \tag{2.10}$$

where $0 < u := 1 + 7^{-n} - 6^{-n} < 1$. Now $A_1 \subset \mathbb{R}^n \setminus (\cup_{i=1}^{k_1} B_i)$ and therefore we can find an open set U_1 such that

$$A_1 \subset U_1 \subset \mathbb{R}^n \setminus (\cup_{i=1}^{k_1} B_i)\,,$$
$$\mathcal{L}^n(U_1) \leqslant (1 + 7^{-n})\,\mathcal{L}^n(A_1)\,.$$

Arguing as above there are disjoint balls $B_{k_1+1}, \ldots, B_{k_2}$ in $\mathcal{G}$ such that $B_i \subset U_1$ for $i = k_1 + 1, \ldots, k_2$ and, if

$$A_2 := A_1 \setminus \cup_{i=k_1+1}^{k_2} B_i = A \setminus \cup_{i=1}^{k_2} B_i\,,$$

$$\mathcal{L}^n(A_2) \leqslant u\,\mathcal{L}^n(A_1)\,. \tag{2.11}$$

Thus, from (2.10) and (2.11), it follows that there exists a finite family $\mathcal{F}_2 := \{B_1, \ldots, B_{k_2}\} \subset \mathcal{G}$ of disjoint balls such that $\mathcal{F}_1 \subset \mathcal{F}_2$ and

$$\mathcal{L}^n(A \setminus \cup\mathcal{F}_2) \leqslant u^2\,\mathcal{L}^n(A)\,.$$

After m steps, we have that there exist m finite families $\mathcal{F}_i := \{B_1, \ldots, B_{k_i}\} \subset \mathcal{G}$ $(i = 1, \ldots, m)$ of disjoint balls such that $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m$

$$\mathcal{L}^n(A \setminus \cup\mathcal{F}_m) \leqslant u^m\,\mathcal{L}^n(A)\,. \tag{2.12}$$

If $\mathcal{L}^n(A \setminus \cup\mathcal{F}_m) = 0$ for some m, the procedure stops and we are done. Otherwise we construct an increasing sequence $(\mathcal{F}_m)_m$ of finite disjoint subfamilies of $\mathcal{G}$ such that (2.12) holds for each $m \in \mathbb{N}$. Let us define

$$\mathcal{F} := \cup_{m=1}^{\infty} \mathcal{F}_m\,.$$

then, from (2.12), it follows that

$$\mathcal{L}^n(A \setminus \cup\mathcal{F}) \leqslant \mathcal{L}^n(A \setminus \cup\mathcal{F}_m) \leqslant u^m\,\mathcal{L}^n(A) \quad \forall\, m \in \mathbb{N}\,.$$

Taking the limit as $m \to \infty$ in the previous inequality, we complete the proof.

2nd step: Assume A unbounded. We can write $\mathbb{R}^n = \cup_{i=1}^{\infty} Q_i$ where $(Q_i)_i$ is a sequence of closed cubes of $\mathbb{R}^n$ such that $\mathring{Q}_i \cap \mathring{Q}_j = \emptyset$ if $i \neq j$. Applying the first step to $A \cap \mathring{Q}_i$, for given i, and noticing that

$$\mathcal{L}^n\left(A \setminus \cup_{i=1}^{\infty} \mathring{Q}_i\right) = 0$$

we complete the proof.

□

Remark 2.8. (i) A simple analysis of the proof of Theorem 2.7 yields that it still holds true for a fine cover $\mathcal{G}$ composed of open balls.

(ii) All that we really used of the Lebesgue measure in the proof of Theorem 2.7 was the equality $\mathcal{L}^n(B(x, 5r)) = 5^n\,\mathcal{L}^n(B(x, r))$ in fact only the inequality "$\leqslant$". It is rather straightforward to modify the above proof to see that the theorem remain valid if $\mathcal{L}^n$ is replaced by any Radon measure μ on $\mathbb{R}^n$ such that for some $\tau \in (1, \infty)$,

$$\limsup_{r \to 0} \frac{\mu(B(x, \tau r))}{\mu(B(x, r))} < \infty \quad \mu - \text{a.e. } x \in \mathbb{R}^n\,. \tag{2.13}$$

Moreover, the balls can be replaced by more general families of closed sets and $\mathbb{R}^n$ by more general spaces, see Federer [Fe, 2.8] for example. However, the above theorem is not valid even for all very nice Radon measures on $\mathbb{R}^n$, as the following example shows.

Example 2.9 (Vitali covering property does not hold for all Radon measures in $\mathbb{R}^n$ [Ma]). Let μ be the Radon outer measure in $\mathbb{R}^2$ defined by

$$\mu(E) := \mathcal{L}^1\left(\{x \in \mathbb{R} : \, (x,0) \in E\}\right) \text{ if } E \subset \mathbb{R}^2 .$$

It is easy to see that $\mu = \mathcal{H}^1 \llcorner A$ where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure on $\mathbb{R}^2$ and $A := \{(x,y) \in \mathbb{R}^2 : \, y = 0\}$. The family of balls

$$\mathcal{G} := \{B((x,y),y) : \, 0 < y < \infty\}$$

covers finely A. But for any countable subfamily $\mathcal{F} \subset \mathcal{G}$

$$\mu(A \cap (\cup\mathcal{F})) = 0 .$$

Thus a Vitali covering property like the Lebesgue measure (see Theorem 2.7) **cannot hold for a general Radon measure in** $\mathbb{R}^n$.

Here A touches only the boundaries of the balls of $\mathcal{G}$. By a slight modification, as suggested in [Ma], we could find a family $\mathcal{G}$ such that each point of A is an interior point of arbitrarily small balls of $\mathcal{G}$ and yet the conclusion of Theorem 2.7 fails.

Exercise (suggested by R. Serapioni). Let μ be the the Radon measure as before with $A := \{(x,0) : \, x \in [0,1]\}$.

Let $\mathcal{G}$ be the family of open balls

$$\mathcal{G} := \{U_{x,n} : \, x \in [0,1], \, n \in \mathbb{N}\}$$

with

$$U_{x,n} := U\left(\left(x, \frac{1}{n}\right), r_n\right) \text{ and } r_n := \frac{1}{n} + \alpha\, e^{-n} ,$$

where $\alpha \in (0,1]$ to be fixed later.

(i) $\mathcal{G}$ is a fine cover of A;
(ii) $\mu(A \cap U_{x,n}) \leqslant \sqrt{3\alpha}\, e^{-\frac{n}{2}}$ for each $x \in [0,1]$ and n;
(iii) for a given n, the number of disjoint balls $U_{x,n}$ with $x \in [0,1]$ is at most $n/2$;
(iv) let $\mathcal{F} \subset \mathcal{G}$ be a disjoint, countable family , then, by (ii) and (iii),

$$\mu\left(\cup\mathcal{F}\right) \leqslant \sum_{n=1}^{\infty} \frac{n}{2}\sqrt{3\alpha}\, e^{-\frac{n}{2}} \leqslant \sqrt{\alpha} \sum_{n=1}^{\infty} n\, e^{-\frac{n}{2}} < \frac{1}{2}$$

for α small enough.

Thus, by (iv), for any disjoint subfamily $\mathcal{F} \subset \mathcal{G}$,

$$\mu\left(A \setminus (\cup\mathcal{F})\right) = \mu(A) - \mu\left(A \cap (\cup\mathcal{F})\right) \geqslant \mu(A) - \mu\left(\cup\mathcal{F}\right) \geqslant \frac{1}{2} .$$

However, if we should require that each point of A is the centre (in fact, not too far from the centre would be enough) of arbitrarily small balls of $\mathcal{G}$ we would get the conclusion of Theorem 2.7. Next we shall develop a covering theorem of this type.

Besicovitch's covering theorem and Radon measures on $\mathbb{R}^n$.

Again we shall first introduce a theorem called Besicovitch's covering theorem, which originated from Besicovitch [Be1] and [Be2].

Besicovitch's covering theorem 2.10. *There are integers $P(n)$and $Q(n)$ depending only on n with the following properties. Let A be a bounded subset of $\mathbb{R}^n$, and let $\mathcal{G}$ be a family of closed balls such that each point of A is the centre of some ball of $\mathcal{G}$.*

(i) *There is a finite or countable subfamily $\mathcal{F} \subset \mathcal{G}$ which covers A and every point of $\mathbb{R}^n$ belongs to at most $P(n)$ balls of $\mathcal{F}$, that is,*

$$\chi_A \leqslant \sum_{B\in\mathcal{F}} \chi_B \leqslant P(n) .$$

(ii) *There are subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_{Q(n)} \subset \mathcal{G}$ covering A such that each $\mathcal{F}_i$ is disjoint, that is,*

$$A \subset \cup_{i=1}^{Q(n)} (\cup \mathcal{F}_i)$$

and

$$B \cap B' = \emptyset \text{ for } B, B' \in \mathcal{F}_i \text{ with } B \neq B' .$$

Proof. See [Ma, Theorem 2.7]. □

We can now easily establish a Vitali-type covering theorem for arbitrary Radon measures on $\mathbb{R}^n$.

Theorem 2.11 (Vitali covering property for Radon measures)**.** *Let φ be a Radon o. m. in $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ (even not φ-measurable) and $\mathcal{G}$ a family of closed balls. Assume that $\mathcal{G}$ is a cover of A and*

$$\inf \{ r : B(x,r) \in \mathcal{G} \} = 0 \quad \forall x \in A . \tag{2.14}$$

Then there is a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that

$$\varphi (A \setminus \cup\mathcal{F}) = 0 .$$

Proof. **1st step:** Suppose first A is bounded and we may assume $0 < \varphi(A) < \infty$. By Theorem 1.14, there is an open set U_0 such that $A \subset U_0$ and

$$\varphi(U_0) < (1 + (4Q(n))^{-1})\varphi(A) ,$$

where $Q(n)$ is as in Besicovitch's covering theorem 2.10 . By that theorem we can find subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_{Q(n)} \subset \mathcal{G}$ such that each $\mathcal{F}_i$ is disjoint and

$$A \subset \cup_{i=1}^{Q(n)} \cup \mathcal{F}_i \subset U_0 .$$

Then

$$\varphi(A) \leqslant \sum_{i=1}^{Q(n)} \varphi (\cup\mathcal{F}_i)$$

and consequently there is an i_* with

$$\varphi(A) < Q(n)\varphi (\cup\mathcal{F}_{i_*}) .$$

Further, for some finite subfamily $\mathcal{F}'_{i_*} \subset \mathcal{F}_{i_*}$ we have

$$\varphi(A) < 2Q(n)\varphi \left(\cup\mathcal{F}'_{i_*}\right) .$$

Letting

$$A_1 = A \setminus \cup\mathcal{F}'_{i_*},$$

we get with $u = 1 - \frac{1}{4}Q(n)^{-1} < 1$

$$\varphi(A_1) < \varphi\left(U_0 \setminus \cup\mathcal{F}'_{i_*}\right) = \varphi(U_0) - \varphi\left(\cup\mathcal{F}'_{i_*}\right)$$
$$< \left(1 + \frac{1}{4}Q(n)^{-1} - \frac{1}{2}Q(n)^{-1}\right)\varphi(A) = u\,\varphi(A)$$

We can now continue by the same principle as in the proof of Theorem 2.7.

2nd step: Assume A unbounded. We may modify the last step of the proof of Theorem 2.7 making use of the fact that, by Lemma 1.112, $\varphi(H)$ can be positive for at most countably many parallel hyperplanes H. More precisely, if $i = 1, \ldots, n$ and $t \in \mathbb{R}$, denote by

$$H_t^{(i)} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = t\},$$

and

$$N_i := \left\{t \in \mathbb{R} : \varphi\left(H_t^{(i)}\right) > 0\right\}.$$

Then N_i is at most countable for each $i = 1, \ldots, n$. Therefore, for each $i = 1, \ldots, n$, there exists a sequence $(t_k^{(i)})_{k\in\mathbb{Z}} \subset \mathbb{R}$ such that

$$k \leqslant t_k^{(i)} < k + \frac{1}{2} \text{ and } \varphi\left(H^{(i)}_{t_k^{(i)}}\right) = 0 \quad \forall\, k \in \mathbb{Z}.$$

Let us define the closed rectangles

$$Q_k := \Pi_{i=1}^n [t_{k-1}^{(i)}, t_k^{(i)}] \quad \text{if } k \in \mathbb{Z}.$$

Then, it is easy to see that

$$\mathbb{R}^n = \cup_{k\in\mathbb{Z}} Q_k,\ \overset{\circ}{Q}_k \cap \overset{\circ}{Q}_{k'} = \emptyset \text{ if } k \neq k'$$

and

$$\varphi\left(\mathbb{R}^n \setminus \cup_{k\in\mathbb{Z}} \overset{\circ}{Q}_k\right) = 0.$$

Applying the first step to $A \cap \overset{\circ}{Q}_k$ we complete the proof. □

Remark 2.12. The above theorem still holds true for families of open balls if φ is the Lebesgue measure (in this case it reduces to the classical Vitali covering theorem 2.7); if φ is a general Radon measure, further conditions have to be imposed: for instance, we may require that for every $x \in A$ and $\epsilon > 0$ the cardinality of the balls of $\mathcal{G}$ centred at x with radius less than $\epsilon > 0$ is more than countable, or that this property fails for a φ-negligible set of points. In this case, in fact, it is possible to select only those balls B such that $\varphi(\partial B) = 0$ (thus getting again a fine cover) and apply Theorem 2.11 to the cover given by the closure of the selected balls. However. it is interesting to note that Theorem 2.11 does not hold in its full generality for families of open balls, as the next example shows.

Example 2.13. Let $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ and consider, respectively, the Radon (outer) measure φ on $\mathbb{R}$ and family of open intervals $\mathcal{G}$ defined as

$$\varphi := \sum_{i=1}^{\infty} \frac{1}{2^i}\delta_{q_i}, \quad \mathcal{G} := \{(a,b) : a, b \in \mathbb{Q}, a < b\}.$$

Then $\mathcal{G}$ can be also meant as a family of open balls in $\mathbb{R}$, which finely covers $\mathbb{Q}$. On the other hand, for each countable, disjoint subfamily $\mathcal{F} \subset \mathcal{G}$ it follows that

$$\varphi(\mathbb{Q} \setminus \cup\mathcal{F}) \geqslant \varphi(\cup_{U\in\mathcal{F}}\partial U) > 0.$$

Historical notes:([dG, I.3]) The most classical covering theorem in differentiation theory is that of Vitali [Vitali2], which has traditionally been the tool to obtain the Lebesgue differentiation theorem in $\mathbb{R}^n$. In its original form the theorem of Vitali refers to closed cubic intervals and the Lebesgue measure.

Later on Lebesgue [Le2] and others gave it a less rigid geometric form replacing cubes by other sets "regular" with respect to cubes, keeping always the restriction to the Lebesgue measure. This restriction originates in the type of proof of the theorem, essentially that given by Banach [Ba], which requires that homothetic sets have comparable measures. Also Caratheodory's proof [C] is based on this property although it is a little different.

Besicovitch [Be1, Be2] and A.P. Morse [Mor] were the first in obtaining similar covering lemmas for more general measures in order to prove differentiability properties analogous to that of the Lebesgue theorem.

2.2. **Derivatives of Radon measures on $\mathbb{R}^n$. Lebesgue-Besicovitch differentiation theorem for Radon measures on $\mathbb{R}^n$.** In this section the environment metric space will be $X = \mathbb{R}^n$ and $\mathcal{M} = \mathcal{B}(\mathbb{R}^n)$.

Notation: If $x \in \mathbb{R}^n$, $r > 0$, ν and μ are positive Radon measures on $\mathbb{R}^n$, then we interpret

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} := \begin{cases} \dfrac{\nu(B(x,r))}{\mu(B(x,r))} & \text{if } x \in \operatorname{spt}(\mu) \\ \infty & \text{if } x \in \mathbb{R}^n \setminus \operatorname{spt}(\mu) \end{cases} ;$$

if $f \in L^1_{\rm loc}(\mathbb{R}^n, \mu)$, $A \subset \mathbb{R}^n$ bounded with $\mu(A) > 0$, $⨍_A f\, d\mu := \dfrac{\int_A f\, d\mu}{\mu(A)}$,

Derivatives of positive Radon measures.

Let ν and ν be a positive Radon measure on $\mathbb{R}^n$ and assume that $\nu << \mu$. Then, from the Radon-Nikodym theorem for Radon measures, there exists $w \in L^1_{loc}(\mathbb{R}^n, \mu)$, $w \geqslant 0$ such that $w = \dfrac{d\nu}{d\mu}$, that is,

$$\nu(E) := \int_E w\, d\mu \quad \forall\, E \in \mathcal{B}(\mathbb{R}^n).$$

Question:

$$\text{(D)} \qquad \exists \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = \lim_{r\to 0} ⨍_{B(x,r)} w(y)\, d\mu(y) = w(x) \quad \mu - \text{a.e. } x \in \mathbb{R}^n\,?$$

Definition 2.14. *Let ν and μ be positive Radon measures on $\mathbb{R}^n$.*

(i) *The upper and lower derivatives of ν with respect to μ at a point $x \in \mathbb{R}^n$ are defined by*

$$\overline{D}_\mu \nu(x) = \limsup_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \in [0, +\infty]\,; \tag{2.15}$$

$$\underline{D}_\mu \nu(x) = \liminf_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \in [0, +\infty]\,. \tag{2.16}$$

(ii) *At a point $x \in \mathbb{R}^n$ where the limit exists, we define the derivative of ν with respect to μ by*

$$D_\mu \nu(x) = \overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x)\,. \tag{2.17}$$

The basic differentiation result for positive Radon measures on $\mathbb{R}^n$ is contained in the following.

Theorem 2.15 (Differentation for positive Radon measures)**.** *Let ν and μ be positive Radon measures on $\mathbb{R}^n$.*

(i) *The derivative $D_\mu \nu(x)$ exists and is finite (that is $D_\mu \nu(x) \in [0, \infty)$) for μ-a.e. $x \in \mathbb{R}^n$.*

(ii) *The function $D_\mu \nu : \mathbb{R}^n \to [0, +\infty]$ is Borel measurable, by defining $D_\mu \nu = \infty$ on the possible μ-negligible set where it does not exist.*

(iii) *Let*

$$A := \{x \in \mathbb{R}^n : \exists\, D_\mu \nu(x) \in [0, \infty)\}\ . \tag{2.18}$$

For all Borel sets $B \subset \mathbb{R}^n$

$$\int_B D_\mu \nu \, d\mu = \nu(A \cap B) \leqslant \nu(B)\,, \tag{2.19}$$

with equality if $\nu << \mu$. In this case

$$D_\mu \nu(x) = \frac{d\nu}{d\mu}(x) = \frac{d\nu_{ac}}{d\mu}(x) \quad \mu\text{-a.e. } x \in \mathbb{R}^n\,.$$

denoting $\dfrac{d\lambda}{d\mu}$ the Radon-Nikodym derivative of λ with respect to μ.

(iv) *$\nu << \mu$ if and only if $\underline{D}_\mu \nu(x) < \infty$ ν-a.e. $x \in \mathbb{R}^n$.*

As a corollary we obtain the following fundamental result.

Theorem 2.16 (Lebesgue-Besicovitch differentiation theorem)**.** *Let μ be a positive Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\exists \lim_{r \to 0} \fint_{B(x,r)} f(y)\, d\mu(y) = f(x) \quad \mu - a.e.\ x \in \mathbb{R}^n\,,$$

that is, by definition, there exists a μ-negligible set $N \subset \mathbb{R}^n$ (i.e. $\mu(N) = 0$) such that

$$\exists \lim_{r \to 0} \fint_{B(x,r)} f(y)\, d\mu(y) = f(x) \quad \forall\, x \in \mathbb{R}^n \setminus N\,. \tag{$\star$}$$

Remark 2.17. If $f \in L^1_{loc}(\mathbb{R}^n\mu)$, denote by

$$[f] := \left\{ g : \mathbb{R}^n \to \overline{\mathbb{R}} : g \text{ measurable}, \ g = f \ \mu\text{- a.e. in } \mathbb{R}^n \right\} .$$

Then the previous theorem actually states that there exists a μ-negligible set $N(f)$ such that

$$\exists \lim_{r\to 0} \fint_{B(x,r)} g(y)\, d\mu(y) = \tilde{f}(x) \quad \forall\, x \in \mathbb{R}^n \setminus N(f),\ g \in [f]\,,$$

and

$$\tilde{f}(x) = f(x) \quad \forall\, x \in \mathbb{R}^n \setminus N(f)\,.$$

Thus the limit in $(\star)$ provides a way to define the value of f at x that is independent of the choice of representative in the equivalence class of f. Notice also that $(\star)$ can be written as

$$\exists \lim_{r\to 0} \fint_{B(x,r)} (f(y) - f(x))\, d\mu(y) = 0 \quad \mu - \text{a.e. } x \in \mathbb{R}^n\,.$$

Lebesgue-Besicovitch differentiation theorem yields at once the following result about the density of a point with respect to a set.

Corollary 2.18 (Density of a set)**.** *Let μ be a positive Radon measure on $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ be a measurable set. Then*

$$\exists \lim_{r\to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } x \in E \\ 0 & \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n \setminus E \end{cases},$$

that is, μ-a.e. $x \in E$ is a point of density 1 for E *and μ-a.e. $x \in \mathbb{R}^n \setminus E$ is* a point of density 0 for E.

Proof of Theorem 2.16. It is not restrictive to assume that $f \geqslant 0$. Otherwise we can decompose $f = f^+ - f^-$ and to apply the same argument to f^+ and f^-.

Let us define the Radon measure

$$\nu(B) := \int_B f\, d\mu \quad \text{if } B \in \mathcal{B}(\mathbb{R}^n)\,.$$

By applying Theorem 2.15 (iii) we get that

$$\int_B D_\mu \nu(x)\, d\mu(x) = \nu(B) = \int_B f\, d\mu(x) \quad \forall B \in \mathcal{B}(\mathbb{R}^n)\,,$$

whence

$$D_\mu \nu(x) = f(x) \quad \mu\text{-a.e. } x \in \mathbb{R}^n\,.$$

□

Proof of Theorem 2.1. For each $\varrho \in \mathbb{Q}$, apply Theorem 2.16 to conclude that there is a set $N_\varrho \subset \mathbb{R}^n$ with $\mu(N_\varrho) = 0$ such that

$$\lim_{r\to 0} \fint_{B(x,r)} |f(y) - \varrho|\, d\mu(y) = |f(x) - \varrho| \quad \forall\, x \in \mathbb{R}^n \setminus N_\varrho\,. \tag{2.20}$$

Thus, with $N = \cup_{\varrho\in\mathbb{Q}} N_\varrho$, we have $\mu(N) = 0$. Moreover, for $x \in \mathbb{R}^n \setminus N$, $\varrho \in \mathbb{Q}$, since

$$|f(y) - f(x)| \leqslant |f(y) - \varrho| + |\varrho - f(x)|\,,$$

(2.20) implies that

$$\limsup_{r\to 0} \fint_{B(x,r)} |f(y)-f(x)|\,d\mu(y) \leqslant 2\,|f(x)-\varrho| \quad \forall\, x\in\mathbb{R}^n\setminus N,\ \varrho\in\mathbb{Q}\,.$$

Since

$$\inf\{|f(x)-\varrho| : \varrho\in\mathbb{Q}\} = 0\,,$$

the proof is complete. □

The proof of Theorem 2.15 relies on two preliminary results: the Vitali covering property for Radon measures on $\mathbb{R}^n$ (see Theorem 2.11), which yields the following comparison between ν and μ provided a pointwise estimates on the derivatives of ν with respect to μ.

Lemma 2.19 (density estimates for differentiation of measures)**.** *Let ν and μ be positive Radon measures on $\mathbb{R}^n$, let $A\subset\mathbb{R}^n$ be a Borel set and $0<t<\infty$.*

(i) *If $\underline{D}_\mu\nu(x)\leqslant t$ for each $x\in A$, then $\nu(A)\leqslant t\,\mu(A)$;*

(ii) *If $\overline{D}_\mu\nu(x)\geqslant t$ for each $x\in A$, then $\nu(A)\geqslant t\,\mu(A)$;*

(iii) *$\underline{D}_\mu\nu$, $\overline{D}_\mu\nu : \mathbb{R}^n\to[0,\infty]$ are Borel measurable.*

In particular

$$\mu(\{x\in\mathbb{R}^n : \overline{D}_\mu\nu(x)=\infty\})=0\,. \tag{2.21}$$

Proof. (i) It is not restrictive to assume that A is bounded. By the approximation of Borel measures by open and closed sets (see Corollary 1.12), for every $\varepsilon>0$ there exists a bounded open set $U\supset A$ such that $\mu(U)<\mu(A)+\varepsilon$. Moreover, by definition of lim inf, for each $x\in A$ and $\epsilon>0$, there exists a sequence of positive real numbers $(r_h)_h$ such that

$$\lim_{h\to\infty} r_h = 0 \quad\text{and}\quad \nu(B(x,r_h))\leqslant (t+\epsilon)\,\mu(B(x,r_h)),\ B(x,r_h)\subset U \quad \forall h. \tag{2.22}$$

Fix $\varepsilon>0$ and let us define

$$\mathcal{G} := \Big\{B(x,r) : x\in A,\ r\in(0,+\infty)\text{ with } B(x,r)\subset U,\ \nu(B(x,r))\leqslant (t+\varepsilon)\,\mu(B(x,r))\Big\},\,.$$

From (2.22), it is easy to see that $\mathcal{G}$ is a family of closed balls which covers A and satisfies (2.14). Applying the Vitali covering property for Radon measures on $\mathbb{R}^n$ (see Theorem 2.11), there exists a countable subfamily

$$\mathcal{F} := \{B_i : i\in\mathbb{N}\}\subset\mathcal{G}$$

such that

$$B_i\cap B_j=\emptyset \text{ if } i\neq j \quad\text{and } \nu\left(A\setminus\cup_{i=1}^\infty B_i\right)=0\,.$$

Thus

$$\begin{aligned}\nu(A) &\leqslant \nu(\cup_{i=1}^\infty B_i)=\sum_{i=1}^\infty \nu(B_i)<(t+\varepsilon)\sum_{i=1}^\infty\mu(B_i)=(t+\varepsilon)\,\mu(\cup_{i=1}^\infty B_i)\\ &\leqslant (t+\varepsilon)\,\mu(U)<(t+\varepsilon)\,(\mu(A)+\varepsilon)\,.\end{aligned} \tag{2.23}$$

Letting $\varepsilon\to 0$, the desired inequality follows.

(ii) It is not restrictive to assume that A is bounded. By the approximation of Borel measures by open and closed sets, for every $\varepsilon>0$ there exists an open set

$U \supset A$ such that $\nu(U) < \nu(A) + \varepsilon$. Moreover, by definition of lim sup, for each $x \in A$ there exists a sequence of positive real numbers $(r_h)_h$ such that

$$\text{(2.24)} \qquad \lim_{h\to\infty} r_h = 0 \quad \text{and} \quad \nu(B(x, r_h)) \geqslant (t - \epsilon)\, \mu(B(x, r_h)),\ B(x, r_h) \subset U \quad \forall h, .$$

Fix $\varepsilon > 0$ and let us define

$$\mathcal{G} := \Big\{ B(x, r) : x \in A,\ r \in (0, +\infty) \text{ with } B(x, r) \subset U,\ \nu(B(x, r)) \geqslant (t{-}\varepsilon)\mu(B(x, r)) \Big\}.$$

From (2.24), it is easy to see that $\mathcal{G}$ is a family of closed balls which covers A and satisfies (2.14). Applying the Vitali covering property for Radon measures on $\mathbb{R}^n$ (see Theorem 2.11), there exists a countable subfamily

$$\mathcal{F} := \{B_i : i \in \mathbb{N}\} \subset \mathcal{G}.$$

such that

$$\text{(2.25)} \qquad B_i \cap B_j = \emptyset \text{ if } i \neq j \quad \text{and } \mu\left(A \setminus \cup_{i=1}^\infty B_i\right) = 0.$$

Thus, by (2.25) and (2.24),

$$\text{(2.26)} \qquad (t - \varepsilon)\, \mu(A) \leqslant (t - \varepsilon)\, \mu(\cup_{i=1}^\infty B_i) = \sum_{i=1}^\infty (t - \varepsilon)\, \mu(B_i) \leqslant \sum_{i=1}^\infty \nu(B_i)$$
$$= \nu(\cup_{i=1}^\infty B_i) \leqslant \nu(U) < (\nu(A) + \varepsilon).$$

Letting $\varepsilon \to 0$, the desired inequality follows.

(iii) Fix $r > 0$ and let $g_r : \mathbb{R}^n \to [0, \infty]$ be the function

$$g_r(x) := \frac{\nu(B(x, r))}{\mu(B(x, r))} \quad \text{if } x \in \mathbb{R}^n.$$

Exercise: Prove that:

(a) given $r > 0$, the function $\mathbb{R}^n \ni x \mapsto \nu(B(x, r))$ (and also the function $\mathbb{R}^n \ni x \mapsto \mu(B(x, r))$) is upper semicontinuous, that is

$$\nu(B(x, r)) \geqslant \limsup_{y\to x} \nu(B(y, r)) \quad \forall x \in \mathbb{R}^n;$$

(b) given $r > 0$, the function $g_r : \mathbb{R}^n \to [0, \infty]$ is Borel measurable;

(c) given $x \in \mathbb{R}^n$, the function $(0, \infty) \ni r \mapsto \nu(B(x, r))$ (and also the function $(0, \infty) \ni r \mapsto \mu(B(x, r))$) is right-continuous, that is

$$\lim_{s\to r^+} \nu(B(x, s)) = \nu(B(x, r)) \quad \forall\, r \in (0, \infty);$$

(d) $\underline{D}_\mu\nu(x) = \lim_{h\to\infty}\left(\inf_{r\in(0,1/h)\cap\mathbb{Q}} g_r(x)\right)$, $\overline{D}_\mu\nu(x) = \lim_{h\to\infty}\left(\sup_{r\in(0,1/h)\cap\mathbb{Q}} g_r(x)\right)$ for each $x \in \mathbb{R}^n$.

From (d) and (b), it follows that both $\underline{D}_\mu$ and $\overline{D}_\mu$ are Borel measurable.

Let us now prove (2.21). For each $h \in \mathbb{N}$ let

$$E_h := \{x \in B(h) : \overline{D}_\mu\nu(x) = \infty\}.$$

By previous claim (ii), it follows that, for fixed h,

$$\mu(E_h) \leqslant \frac{\nu(B(h))}{t} \quad \forall\, t \in [1, \infty).$$

Since $\nu(B(h)) < \infty$, taking the limit as $t \to \infty$ in the previoua inequality, we get that

$$\mu(E_h) = 0 \quad \forall\, h \in \mathbb{N}$$

and then (2.21) follows. □

Proof of Theorem 2.15. (i) For $0 < R < \infty$, $0 < s < t < \infty$, let

$$A_{s,t,R} := \left\{x \in B(0,R) : \underline{D}_\mu \nu(x) \leqslant s < t < \overline{D}_\mu \nu(x) \right\} ,$$

$$A_{t,R} := \left\{x \in B(0,R) : t \leqslant \overline{D}_\mu \nu(x) \right\} .$$

Then $A_{s,t,R}$ and $A_{t,R}$ are Borel sets and, by Lemma 2.19,

$$t\, \mu(A_{s,t,R}) \leqslant \nu(A_{s,t,R}) \leqslant s\, \mu(A_{s,t,R}) < \infty ,$$

$$u\, \mu(A_{u,R}) \leqslant \nu(A_{u,R}) \leqslant \nu(B(0,R)) < \infty .$$

These inequalities yield

$$\mu(A_{s,t,R}) = 0 \quad \forall\, 0 < s < t < \infty , \tag{2.27}$$

and

$$\mu(\cap_{u>0} A_{u,R}) = \lim_{u\to\infty} \mu(A_{u,R}) = 0 . \tag{2.28}$$

Notice that

$$\cap_{u>0} A_{u,R} = \cap_{n\in\mathbb{N}} A_{n,R} ,$$

and the $\cap_{u>0} A_{u,R}$ is a Borel set.

Let denote by N_1 the set of points $x \in \mathbb{R}^n$ such that $\nexists\, D_\mu \nu(x)$ or $\overline{D}_\mu \nu(x) = \infty$.

Exercise: Prove that

$$N_1 = \cup \left\{A_{s,t,R} : 0 < s < t < \infty,\ s,t \in \mathbb{Q}, R > 0, R \in \mathbb{Q} \right\}$$
$$\cup \left\{\cap_{u>0} A_{u,R} : R > 0, R \in \mathbb{Q} \right\} .$$

Then, from (2.27) and (2.28), it follows that $\mu(N_1) = 0$, which settles (i).

(ii) By Lemma 2.19 (iii), both $\underline{D}_\mu \nu$ and $\overline{D}_\mu \nu$ are Borel measurable functions. Thus, by definition, $D_\mu \nu(x)$ is a Borel measurable function, too.

(iii) Let A be the set defined in (2.18) and let $N_2 := \{x \in \mathbb{R}^n : \exists D_\mu \nu(x) = 0\}$. Observe that

$$\mathbb{R}^n \setminus A \subseteq N_1 . \tag{2.29}$$

Let us begin to prove that

$$\nu(N_2) = 0 . \tag{2.30}$$

For $0 < R < \infty$, $0 < s < \infty$, let

$$A^*_{s,R} := \left\{x \in B(0,R) : D_\mu \nu(x) \leqslant s \right\} .$$

Then, by Lemma 2.19 (i), for given $R > 0$, it follows that

$$\nu(N_2 \cap B(0,R)) \leqslant \nu(A^*_{\epsilon,R}) \leqslant \epsilon\, \mu(A^*_{\epsilon,R}) \leqslant \epsilon\, \mu(B(0,R)) \quad \forall\, \epsilon > 0 .$$

Letting $\epsilon \to 0$ in the previous inequality, we get

$$\nu(N_2 \cap B(0,R)) = 0 \quad \forall R > 0 ,$$

which establishes (2.30). Let us now prove the identity in (2.19). This amounts to prove the two inequalities

$$\int_B D_\mu\nu(x)\,d\mu(x) \leqslant \nu(A\cap B) \quad \forall\, B \in \mathcal{B}(\mathbb{R}^n)\,, \tag{2.31}$$

$$\int_B D_\mu\nu(x)\,d\mu(x) \geqslant \nu(A\cap B) \quad \forall\, B \in \mathcal{B}(\mathbb{R}^n)\,. \tag{2.32}$$

Fix $B \in \mathcal{B}(\mathbb{R}^n)$ and choose $1 < t < \infty$, let

$$B_k := \left\{x \in B : t^k \leqslant D_\mu\nu(x) < t^{k+1}\right\} \quad k \in \mathbb{Z}\,,$$

and notice that

$$(A \setminus N_2) \cap B = \cup_{k\in\mathbb{Z}} B_k \quad B_k \cap B_{k'} = \emptyset \text{ if } k \neq k'\,, \tag{2.33}$$

where A is the set defined in (2.18). By (2.21), (2.28), Lemma 2.19 (i) and (2.30),

$$\int_B D_\mu\nu(x)\,d\mu(x) = \int_{A\cap B} D_\mu\nu(x)\,d\mu(x) = \sum_{k=-\infty}^{\infty}\int_{B_k} D_\mu\nu(x)\,d\mu(x) \leqslant \sum_{k=-\infty}^{\infty} t^{k+1}\,\mu(B_k)$$
$$\leqslant t\sum_{k=-\infty}^{\infty}\nu(B_k) \leqslant t\,\nu((A\setminus N_2)\cap B) = t\,\nu(A\cap B)\,.$$

Letting $t \to 1^+$ in the previous inequality, we establish (2.31). Let us now show (2.32). Choose $0 < t < 1$, let

$$B_k := \left\{x \in B : t^{k+1} \leqslant D_\mu\nu(x) < t^{k}\right\} \quad k \in \mathbb{Z}\,,$$

an notice that (2.33) still holds. Arguing as before, we get

$$\int_B D_\mu\nu(x)\,d\mu(x) = \int_{A\cap B} D_\mu\nu(x)\,d\mu(x) = \sum_{k=-\infty}^{\infty}\int_{B_k} D_\mu\nu(x)\,dx \geqslant \sum_{k=-\infty}^{\infty} t^{k+1}\,\mu(B_k)$$
$$\geqslant t\sum_{k=-\infty}^{\infty}\nu(B_k) = t\,\nu((A\setminus N_2)\cap B) = t\,\nu(A\cap B)\,.$$

Letting $t \to 1^-$, in the previous inequality, we get (2.32) and then the equality in (2.19). The inequality in (2.19) trivially follows by the monotonicity of ν.

If $\nu << \mu$, we have to prove that

$$\nu(A\cap B) = \nu(B) \quad \forall\, B \in \mathcal{B}(\mathbb{R}^n)\,.$$

We need only to show that

$$\nu(\mathbb{R}^n \setminus A) = 0\,.$$

By (2.29), we have only to prove that

$$\nu(N_1) = 0\,. \tag{2.34}$$

By previous point (i), and because of $\nu << \mu$, (2.34) follows.

(iv) If $\nu << \mu$, by previous claim (i), $\underline{D}_\mu\nu(x)$ μ-a.e. $x \in \mathbb{R}^n$ and then $\underline{D}_\mu\nu(x)$ ν-a.e. $x \in \mathbb{R}^n$, too. Suppose now that $\underline{D}_\mu\nu(x) < \infty$ ν-a.e. $x \in \mathbb{R}^n$ and let $B \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(B) = 0$. Lemma 2.19 (i) gives

$$\nu\left(\left\{x \in B : \underline{D}_\mu\nu(x) \leqslant h\right\}\right) \leqslant h\,\mu(B) = 0 \quad \forall\, h \in \mathbb{N}\,.$$

Therefore, since $\nu(\{x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) = \infty\}) = 0$,

$$\begin{aligned}\nu(B) &= \nu\left(\cup_{h=1}^{\infty}\left\{x \in B : \underline{D}_\mu \nu(x) \leqslant h\right\}\right)\\ &\leqslant \sum_{h=1}^{\infty} \nu\left(\left\{x \in B : \underline{D}_\mu \nu(x) \leqslant h\right\}\right) = 0\,.\end{aligned}$$

□

We are now going to deal with the problem of the Lebesgue decomposition of positive Radon measure ν with respect to a given Radon measure μ, by characterizing the absolutely continuous and singular parts in terms of derivatives of measures.

From Theorem 2.15, the following result immediatly follows.

Theorem 2.20 (Lebesgue decomposition in terms of derivatives of measures)**.** *Let ν and μ be positive Radon measures on $\mathbb{R}^n$. Let ν_{ac} and ν_s denote respectively the absolutely continuous and singular parts of ν in the Lebesgue decomposition with respect to μ. Then, for each Borel set B,*

$$\nu_{ac}(B) = \int_B D_\mu \nu(x)\, d\mu, \quad \nu_s(B) = \nu \llcorner S(B) = \nu(S \cap B)$$

where S is the μ-negligible Borel set

$$S = (\mathbb{R}^n \setminus \operatorname{spt}(\mu)) \cup \{x \in \operatorname{spt}(\mu) : D_\mu \nu(x) = \infty\}$$

Proof. Let us recall that, by definition, $D_\mu \nu(x) = \infty$ either if $\nexists D_\mu \nu(x)$ or if $x \notin \operatorname{spt}(\mu)$ and that $A \subset \operatorname{spt}(\mu)$, where A is the set defined in (2.18). Thus, by Theorem 2.15 (i), $\mu(S) = 0$ and

$$\mathbb{R}^n \setminus S = A\,.$$

Moreover we can write

$$\nu = \nu \llcorner A + \nu \llcorner S \tag{2.35}$$

According to (RN) and by (2.19),

$$\nu \llcorner A = \mu_w << \mu \text{ with } w = D_\mu \nu\,, \tag{2.36}$$

and, since $\nu \llcorner A$ and $\nu \llcorner S$ are mutually singular, by (2.35), (2.36)and the uniqueness of Lebesgue decomposition (see Theorem 1.32), we get the desired conclusion. □

As a byproduct of the previous theorem we get the following result (see also [SC, Lemma 1.51]).

Corollary 2.21. *Let ν and μ be two positive Radon measures on $\mathbb{R}^n$ that are (mutually) singular, that is, $\nu \perp \mu$. Then*

$$\exists\, D_\mu \nu(x) = 0 \quad \mu\text{-a.e. } x \in \mathbb{R}^n\,.$$

Proof. From Lebesgue decomposition theorem 1.32, it follows that $\nu_{ac} \equiv 0$ and $\nu = \nu_s$. Thus, from Theorem 2.20, it follows that

$$0 = \nu_{ac}(B) = \int_B D_\mu \nu\, d\mu \text{ for each } B \in \mathcal{B}(\mathbb{R}^n)\,.$$

This implies that $D_\mu \nu(x) = 0$ μ-a.e. $x \in \mathbb{R}^n$. □

In several applications of the differentiation of measures, a differentiation result with respect to a more general class of sets rather than balls could very useful as, for instance, the case of the Lebesgue differentiation theorem for monotone functions (see [GZ, SC]). Therefore we are going to show that a more general class of sets can be considered in the derivative of measures, instead of balls.

Definition 2.22. *Let μ be a positive Radon measure on $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. A sequence of Borel sets $(E_h(x))_h \subset \mathcal{B}(\mathbb{R}^n)$ is called a (regular) differentiation basis at x for μ provided there is $\alpha_x = \alpha(x) > 0$ with the following properties:*

(i) *there exists a sequence of balls $(B(x, r_h))_h$ with $r_h \to 0$ such that*

$$E_h(x) \subset B(x, r_h) \quad \forall\, h\,;$$

(ii) $\mu(E_h(x)) \geqslant \alpha_x\, \mu(B(x, r_h)) \quad \forall\, h\,.$

Theorem 2.23 (Differentiation for positive Radon measures with respect to a differentiation basis)**.** *Let ν and μ be postive Radon measures on $\mathbb{R}^n$, then there exists*

$$\lim_{h\to\infty} \frac{\nu(E_h(x))}{\mu(E_h(x))} = \frac{d\nu_{ac}}{d\mu}(x) = D_\mu \nu_{ac}(x) \quad \mu\text{-a.e. } x \in \operatorname{spt}(\mu)\,,$$

whenever $(E_h(x))_h$ is a differentiation basis of μ at x.

Proof. Recall that, by Corollary 2.21, Theorems 2.20 and 2.15 (i),

$$\nu = \nu_{ac} + \nu_s \tag{2.37}$$

with $\nu_{ac} << \mu$ and $\nu_s \perp \mu$,

$$\exists \lim_{r\to 0} \frac{\nu_s(B(x,r))}{\mu(B(x,r))} = 0 \quad \mu\text{-a.e. } x \in \operatorname{spt}(\mu)\,, \tag{2.38}$$

$$\exists \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = \lim_{r\to 0} \frac{\nu_{ac}(B(x,r))}{\mu(B(x,r))} = \frac{d\nu_{ac}}{d\mu}(x) := w(x)\, \mu\text{-a.e. } x \in \operatorname{spt}(\mu)\,. \tag{2.39}$$

Let $x \in \operatorname{spt}(\mu)$ be such that (2.38) holds and let $(E_h(x))_h$ be a differentiation basis at x of μ. Then

$$0 \leqslant \frac{\nu_s(E_h(x))}{\mu(E_h(x))} \leqslant \frac{1}{\alpha_x} \frac{\nu_s(B(x, r_h))}{\mu(B(x, r_h))}\,,$$

and, from (2.38), it follows that

$$\exists \lim_{h\to\infty} \frac{\nu_s(E_h(x))}{\mu(E_h(x))} = 0\,. \tag{2.40}$$

To conclude the proof we need only to prove that

$$\exists \lim_{h\to\infty} \frac{\nu_{ac}(E_h(x))}{\mu(E_h(x))} = w(x) \tag{2.41}$$

when $x \in \operatorname{spt}(\mu)$ is a Lebesgue point of w (see Definition 2.2).Observe that such a point x satisfies (2.39), too. Thus, (2.37), (2.40) and (2.41) conclude the proof. From Lebesgue points theorem,

$$\exists \lim_{h\to\infty} \fint_{B(x,r_h)} |w(y) - w(x)|\, d\mu(y) = 0 \quad \mu\text{-a.e. } x \in \operatorname{spt}(\mu)\,.$$

Now observe that

$$\fint_{E_h(x)} |w(y) - w(x)| \, d\mu(y) = \frac{1}{\mu(E_h(x))} \int_{E_h(x)} |w(y) - w(x)| \, d\mu(y) \leqslant$$
$$\frac{\mu(B(x, r_h))}{\mu(E_h(x))} \fint_{B(x,r_h)} |w(y) - w(x)| \, d\mu(y) \leqslant \frac{1}{\alpha_x} \fint_{B(x,r_h)} |w(y) - w(x)| \, d\mu(y) \to 0 \, .$$

Thus, it follows that

$$\left| \frac{\nu_{ac}(E_h(x))}{\mu(E_h(x))} - w(x) \right| = \left| \fint_{E_h(x)} w(y) \, d\mu(y) - w(x) \right| =$$
$$\left| \fint_{E_h(x)} (w(y) - w(x)) \, d\mu(y) \right| \leqslant \fint_{E_h(x)} |w(y) - w(x)| \, dy\mu(y) \to 0 \, .$$

Then (2.41) holds. □

Derivatives of vector Radon measures.

We are now going to deal with the problem of the Lebesgue decomposition of vector Radon measure ν with respect to a given Radon measure μ, by characterizing the absolutely continuous and singular parts in terms of derivatives of measures.

Theorem 2.24. *Let ν and μ be respectively a $\mathbb{R}^m$-valued Radon and a positive Radon measures on $\mathbb{R}^n$. Then, for μ-a.e. $x \in \operatorname{spt}(\mu)$,*

$$\exists \, w(x) := \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \in \mathbb{R}^m \, . \tag{2.42}$$

Moreover the Lebsesgue decomposition of ν with respect to μ is given by $\nu = \mu_w + \nu_s$ where $\nu_s(B) = \nu \llcorner S(B) = \nu(S \cap B)$ and S is the μ-negligible Borel set

$$S = (\mathbb{R}^n \setminus \operatorname{spt}(\mu)) \cup \{x \in \operatorname{spt}(\mu) : \, D_\mu|\nu|(x) = \infty\}$$

Proof. Let us recall that, by the polar decomposition of ν with respect to $|\nu|$ (see Corollary 1.53) and the Lebsegue decomposition in terms of derivatives of measures (see Theorem 2.20)

$$\nu(B) = \int_B w_\nu \, d|\nu| \quad \forall \, B \in \mathcal{B}(\mathbb{R}^n) \, , \tag{2.43}$$

$$|\nu|(B) = \int_B D_\mu|\nu| \, d\mu + \nu \llcorner S(B) \quad \forall \, B \in \mathcal{B}(\mathbb{R}^n) \, , \tag{2.44}$$

where $|\nu|$ denotes the total variation of ν and $w_\nu : \mathbb{R}^n \to \mathbb{R}^m$ is a Borel measurable vector function with $|w_\nu| = 1$ $|\nu|$-a.e. in $\mathbb{R}^n$. By combining (2.43) and (2.44), it follows that

$$\nu(B) = \int_B D_\mu|\nu| \, w_\nu \, d\mu + \int_B w_\nu \, d(\nu \llcorner S) \quad \forall \, B \in \mathcal{B}(\mathbb{R}^n) \, . \tag{2.45}$$

In particular, by (2.45) and the the uniqueness of the Lebesgue decomposition of ν with respect to μ (see Theorem 1.52), we get

$$\nu_{ac}(B) = \int_B D_\mu|\nu| \, w_\nu \, d\mu, \;\; \nu_s(B) = \int_B w_\nu \, d(\nu \llcorner S) \quad \forall \, B \in \mathcal{B}(\mathbb{R}^n) \, . \tag{2.46}$$

In addition, since S is μ-negligible, by (2.45) and the Lebesgue-Besicovitch differentiation theorem 2.16, we can infer

$$\exists\, w(x) := \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = D_\mu|\nu|(x)\, w_\nu(x) \in \mathbb{R}^m \quad \mu \text{ a.e. } x \in \operatorname{spt}(\mu)\,. \tag{2.47}$$

From (2.45), (2.46) and (2.47), we get the desired conclusion. □

2.3. **Extensions to metric spaces.**

About Vitali's covering theorem in metric spaces.

Vitali's covering theorem still holds in any separable metric space (see, for instance, [AT, Theorem 2.2.3]).

If (X, d) is a metric space *boundedly compact*, that is all bounded and closed sets of X are compact, a constructive proof of Vitali's covering theorem, whithout the Hausdorff Maximal Principle, can be given (see [Ma, Theorem 2.1]).

If (X, d) is a metric space, using the Hausdorff maximal principle, one can give much more general Vitali's covering type-results ; for example families of balls can be replaced by many other families of sets (see, for instance, [Fe, 2.8.4-6]).

About Vitali's covering property in metric measure spaces.

We call *metric measure space* a structure (X, d, μ) where (X, d) is a metric space and μ is a positive Radon measure on $(X, \mathcal{B}(X))$. Given a metric measure space (X, d, μ) an important issue in the setting of *analysis in metric space* is to know whether *Vitali's covering property holds for the measure* μ, that is, whether Theorem 2.11 still holds replacing metric measure space $(\mathbb{R}^n, |\cdot|, \mu)$ with (X, d, μ). More precisely we say that *Vitali's covering property holds for the metric measure space* (X, d, μ) if, for each $A \subset X$ measurable and bounded, for each family of closed balls $\mathcal{G}$ covering A and satisfying (2.14), there exists a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that $\mu(A \setminus \cup\mathcal{F}) = 0$.

When speaking of a closed ball B in (X, d), it will be understood B that it comes with a fixed center and radius (although these in general are not uniquely determined by B as a set, since neither center nor radius need be unique). Thus $B = B(x, r)$ for some $x \in X$ and some $r > 0$.

Case of the doubling spaces. Vital's covering property holds when the measure μ is supposed to be *doubling* on metric space (X, d), that is we assume that:

- $\mu(X) > 0$;
- $\mu(B(x, r)) < \infty\ \forall\, x \in X, r > 0$;
- there exists a positive constant $C > 0$ such that

$$\mu(B(x, 2r)) \leqslant C\, \mu(B(x, r)) \quad \forall\, x \in X, r > 0 \tag{2.48}$$

(see, for instance, [He, Theorem 1.6]). Indeed condition (2.48) can be weakened in asymptotic doubling condition (2.13) (see [Fe, 2.8]).

Let us also recall that a metric space (X, d) is said to be *doubling* if there exists an integer $C \geqslant 1$ such that each closed ball with radius $r > 0$ can be covered with less than C balls with radius $r/2$. If μ is a doubling measure on a metric space (X, d), then it is easy to see that (X, d) is doubling. On the other hand, not every doubling space carries a doubling measure (see [He, Chap.10]).

Case of the Besicovich covering property. Vitali covering property also holds for a metric measure space (X, d, μ) with metric space (X, d) satisfying the so-called *Besicovitch Covering Property* (BCP) or also with (X, d) doubling metric space satisfying *Weak Besicovitch Covering Property* (WBCP) (see [LeR]).

Definition 2.25 (Besicovitch Covering Property)**.** We say that BCP holds for (X, d) if there exists an integer $P \geqslant 1$ with the following property. Let A be a bounded subset of (X, d) and let $\mathcal{G}$ be family of closed balls in (X, d) such that each point of A is the center of some ball of $\mathcal{G}$. Then there is subfamily $\mathcal{F} \subseteq \mathcal{G}$, at most countable, whose balls cover A and such that every point in X belongs to at most P balls of $\mathcal{F}$, that is,

$$\chi_A \leqslant \sum_{B \in \mathcal{F}} \chi_B \leqslant P .$$

Definition 2.26 (Weak Besicovitch Covering Property)**.** We say that WBCP holds for (X, d) if there exists an integer $N \geqslant 1$ with the following property. Suppose there exist k points $x_i \in X$ and positive numbers $r_i > 0$ $(i = 1, \ldots, k)$ such that

$$x_i \notin B(x_j, r_j) \text{ if } i \neq j \text{ and } \cap_{i=1}^{k} B(x_i, r_i) \neq \emptyset .$$

Then $k \leqslant N$.

The validity of BCP implies the one of WBCP. We stress that there exists metric spaces for which WBCP holds although BCP is not satisfied. However, when the metric is doubling, both covering properties turn out to be equivalent. An exaustive study of this topic is carried in [LeR].

An early study of metric spaces satisfying BCP was carried out by Federer [Fe, 2.8.9]. In particular he proved that BCP holds in compact Riemannian manifolds and in normed vector spaces of finite dimension. It is also well-known that BCP need not hold in sub-Riemannian structures (see, for instance, [LeR]).

It is simple to show that, if a metric space (X, d) satisfies either a doubling or BCP condition, then (X, d) has finite topological dimension (in the sense of Lebesgue covering dimension) [LeR, 8.3].

It is also easy to prove that WBCP need not hold in an infinite dimensional space. For instance, let X be an infinite dimensional Hilbert space and let $B_i := B(e_i, 1)$ $(i \in \mathbb{N})$, where $\{e_i : i \in \mathbb{N}\} \subset X$ is a set of orthonormal vectors. Then $e_i \notin B_j$ if $i \neq j$ and $0 \in \cap_{i=1}^{\infty} B_i$. Moreover there exist metric measure spaces (X, d, μ) with X separable Hilbert space and μ finite measure for which Vitali's covering property does not hold [Ti].

<u>About the Lebesgue differentiation theorem in metric measure spaces.</u>

If μ is a locally finite Borel measure on a metric space (X, d), we say that *the Lebesgue differentiation theorem holds on metric measure space* (X, d, μ) if

$$\exists \lim_{r \to 0} \fint_{B(x,r)} f(y)\, dy = f(x) \quad \mu\text{-a.e. } x \in X, \, \forall f \in L^1_{\mathrm{loc}}(X, \mu) .$$

where

$$L^1_{\rm loc}(X,\mu) := \Big\{ g : X \to \overline{\mathbb{R}} : g \text{ Borel measurable }, \forall\, x \in X\, \exists\, r_x > 0 \text{ such that}$$
$$\int_{B(x,r_x)} |f(y)|\, d\mu(y) < \infty \Big\}\,.$$

It can be proved that the Lebesgue differentiation theorem holds for a metric measure space (X, d, μ) provided that it satisfies Vitali's covering property (see, for instance, [He, Remark 1.13]). In particular, by the previous arguments, the Lebegue differentiation theorem holds either if (X, d, μ) is doubling or (X, d) satisfies BCP.

An interesting characterization of metric measure spaces satisfying the Lebesgue differentiation theorem is due to D. Preiss [Pre]. Following his terminology, if (X, d) is a metric space, we say that *d is finite dimensional on a subset $Y \subset X$* if there exist $C^* \in [1, \infty)$ and $r^* \in (0, \infty]$ with the following property. Suppose there exist k points $x_i \in Y$ and positive numbers $r_i \in (0, r^*)$ $(i = 1, \ldots, k)$ such that

$$x_i \notin B(x_j, r_j) \text{ if } i \neq j \text{ and } \cap_{i=1}^k B(x_i, r_i) \neq \emptyset\,.$$

Then $k \leqslant C^*$. We say *d is σ-finite dimensional* if X can be written as a countable union of subsets on which d is finite dimensional. Note that WBCP holds on (X, d) if and only if d is finite dimensional on X for some constant $C^* \in [1, \infty)$ and with $r^* = \infty$.

Theorem 2.27 ([Pre])**.** *Let (X, d) be a complete separable metric space. The Lebesgue differentiation theorem holds on (X, d) for all locally finite Borel measures if and only if d is σ-finite dimensional.*

An interesting byproduct of the Lebesgue differentiation theorem is the following.

Proposition 2.28. *Let μ and ν be two locally finite Borel measures on a metric space (X, d) and assume that*

(i) *the Lebesgue differentiation theorem holds for both measure spaces (X, d, μ) and (X, d, ν);*
(ii) *$\mu(B(x, r)) = \nu(B(x, r))$ for each $x \in X$, $0 < r < \infty$.*

Then $\mu(B) = \nu(B)$ for each $B \in \mathcal{B}(\mathbb{R}^n)$.

An outline of the proof of Proposition 2.28 will be proposed in the proof of Theorem 3.30.

The Lebesgue differentation theorem need not hold in an infinite dimensional metric space (see [Ti]).

3. An introduction to Hausdorff measures ([AFP, EG, Ma, Mag]).

Motivation: In this section we introduce Hausdorff measures and dimension for measuring the metric size of quite general sets. They will be one of the basic means for studying geometric properties of sets and expressing results that these studies lead to. In particular they are very useful in this study because:

- they measure both regular submanifolds and not regular subsets of $\mathbb{R}^n$, such as fractal sets;
- they do not depend on the parametrization of the submanifold.

3.1. **Carathéodory's construction and definition of Hausdorff measures on a metric space and their elementary properties; Hausdorff dimension.** The main idea is the construction of (outer) measures in $\mathbb{R}^n$ which allow to measures subset of (roughly speaking) "dimension" $m < n$. For instance, in $\mathbb{R}^3$, we are going to define three measures $\mathcal{H}^1$, $\mathcal{H}^2$ and $\mathcal{H}^3$ such that

$$\begin{cases} \mathcal{H}^1(S) = \text{length}(S) & \text{if S is a curve} \\ \mathcal{H}^2(S) = \text{area}(S) & \text{if S is a surface} \\ \mathcal{H}^3(S) = \text{volume}(S) & \text{if S is a ball} \end{cases}.$$

The basic definitions and first results on Hausdorff measures and dimension are due to Carathéodory [C2] in 1914 and Hausdorff [Ha] in 1919 and they can be introduced in the framework of a general metric space (X, d).

We will start with a more general construction, called *Caratheodory's construction.* It will yield also many other measures some of which will be briefly recalled.

Carathéodory's construction

Let (X, d) be a metric space, $\mathcal{F}$ a family of subsets of X and $\zeta : \mathcal{F} \to [0, \infty]$ a given nonnegative evaluation set function. We make the following two assumptions.

(Ca1) For every $\delta \in (0, \infty]$, there is a sequence $(E_i)_i \subset \mathcal{F}$ such that $X = \cup_{i=1}^{\infty} E_i$ and $d(E_i) \leqslant \delta$ for each $i \in \mathbb{N}$.

(Ca2) For every $\delta > 0$, there is $E \in \mathcal{F}$ such that $\zeta(E) \leqslant \delta$ and $d(E) \leqslant \delta$.

For $0 < \delta \leqslant \infty$ and $A \subset X$ we define

$$\psi_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} \zeta(E_i) : A \subset \cup_{i=1}^{\infty} E_i,\ d(E_i) \leqslant \delta,\ E_i \in \mathcal{F} \right\} \tag{3.1}$$

Remark 3.1. Assumption (Ca1) was only introduced to guarantee that such coverings always exist. The role of (Ca2) is to have $\psi_\delta(\emptyset) = 0$. It also allows us to use coverings $(E_i)_{i \in I}$ with I finite or countable without changing value of $\psi_\delta(A)$.

Remark 3.2. It is easy to see that ψ_δ is monotonic and subadditive so that it is an outer measure. Usually it is highly non-additive and not a Borel measure (see Exercise below).

Evidently,

$$\psi_\delta(A) \leqslant \psi_\epsilon(A) \quad \text{whenever } 0 < \epsilon < \delta \leqslant \infty\,.$$

hence we can define

$$\psi = \psi(\mathcal{F}, \zeta)$$

as follows

$$\psi(A) := \lim_{\delta \to 0} \psi_\delta(A) = \sup_{\delta > 0} \psi_\delta(A) \quad \text{for } A \subset X\,. \tag{3.2}$$

The measure-theoretic behaviour of ψ is much better than that of ψ_δ.

Theorem 3.3. (i) *ψ is a Borel outer measure.*

(ii) *If $\mathcal{F} \subset \mathcal{B}(X)$, then ψ is a Borel regular outer measure.*

Proof. Proof. (i) The proof that ψ is an outer measure is straightforward and left to the reader. To show that ψ is a Borel outer measure, we apply Carathéodory criterion (see Theorem 1.5 (iii)). Let $A,\, B \subset X$ with $d(A, B) > 0$. Choose δ with $0 < \delta < d(A, B)/2$. If the sets $E_1,\, E_2, \dots \in \mathcal{F}$ cover $A \cup B$ and satisfy $d(E_i) < \delta$, then none of them can meet both A and B. Hence

$$\begin{aligned}\sum_{i=1}^{\infty} \zeta(E_i) &\geqslant \sum_{A\cap E_i\neq\emptyset} \zeta(E_i) + \sum_{B\cap E_i\neq\emptyset} \zeta(E_i)\\ &\geqslant \psi_\delta(A) + \psi_\delta(B)\,.\end{aligned}$$

Taking the infimum over all such coverings we have $\psi_\delta(A \cup B) \geqslant \psi_\delta(A) + \psi_\delta(B)$. But the opposite inequality holds also as ψ_δ is an outer measure, and so $\psi_\delta(A\cup B) = \psi_\delta(A) + \psi_\delta(B)$. Letting $\delta \to 0$, we obtain $\psi(A \cup B) = \psi(A) + \psi(B)$ as required.

(ii) If $A \subset X$, choose for every $i = 1, 2, \dots$ sets $E_{i,1}, E_{i,2}, \dots \in \mathcal{F}$ such that

$$\begin{aligned}&A \subset \cup_{j=1}^{\infty} E_{i,j},\ d(E_{i,j}) < 1/i \text{ and}\\ &\sum_{j=1}^{\infty} \zeta(E_{i,j}) < \psi_{1/i}(A) + 1/i\,.\end{aligned}$$

Then $B := \cap_{i=1}^{\infty}(\cup_{j=1}^{\infty} E_{i,j})$ is a Borel set such that $A \subset B$ and $\psi(A) = \psi(B)$. Thus ψ is a Borel regular outer measure. □

Hausdorff measures

Let (X, d) be separable metric space, $0 \leqslant s < \infty$, and choose

$$\mathcal{F} := \mathcal{P}(X) = \{E : E \subset X\}\,, \tag{3.3}$$

$$\zeta(E) = \zeta_s(E) := \alpha_s\, d(E)^s \tag{3.4}$$

with the interpretations $0^0 = 1$ and $d(\emptyset)^s = 0$, where α_s is a geometric constant which depends only on s and the environment (X, d) and will be fixed later in the Euclidean context for normalization purposes of constants (see (3.11)).

Exercise: Prove that (Ca1) and (Ca2) are satisfied under assumptions (3.3) and (3.4).

The resulting measure ψ is called the s-dimensional *Hausdorff measure* and denoted by $\mathcal{H}^s$. So

$$\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A)\,. \tag{3.5}$$

and $\mathcal{H}^s_\delta$ is the s-dimensional *Hausdorff pre-measure* defined by

$$\mathcal{H}^s_\delta(A) := \inf\left\{\alpha_s \sum_{i=1}^{\infty} d(E_i)^s : A \subset \cup_{i=1} E_i,\, d(E_i) \leqslant \delta\right\} \tag{3.6}$$

where, $\delta \in (0, \infty]$. Let us observe that $\mathcal{H}^s_\delta : \mathcal{P}(X) \to [0, \infty]$ is an outer measure but is not a Borel outer measure (see [Ma, Chap.4, Ex. 1]).

Exercise: Let $X = \mathbb{R}^n$, $0 < s < \infty$, $0 < \delta \leqslant \infty$. Prove that

(i) $\mathcal{H}^s_\delta : \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$ is increasing, countably subadditive and $\mathcal{H}^s_\delta(\emptyset) = 0$. In particular it is an outer measure.

(ii) $\mathcal{H}^s_\delta$ is neither additive nor a Borel measure. Indeed it can be proved that, if $n \geqslant 2$, $0 \leqslant s \leqslant 1$, $0 < \delta < \infty$ and $U \equiv U_\delta = U(0,\delta/2) = U(\delta/2)$, then

$$\mathcal{H}^s_\delta(U) = \mathcal{H}^s_\delta(\bar{U}) = \mathcal{H}^s_\delta(\partial U)\,.$$

(iii) $\mathcal{H}^s_\delta(A) \leqslant \mathcal{H}^s_\sigma(A)$ for each $0 \leqslant \sigma < \delta \leqslant \infty$, $0 \leqslant s < \infty$.

(*Hint:* (ii) **1st step:** Prove that,

$$\text{if } n = 2 \text{ and } s = 1\,,$$

then

$$\mathcal{H}^1_\delta(\partial U) = \delta\,. \tag{3.7}$$

Indeed, inequality

$$\mathcal{H}^1_\delta(\partial U) \leqslant \delta$$

is trivial by the the definition of $\mathcal{H}^1_\delta$ (recall that $\alpha_1 = 1$). The reverse inequality is less trivial and the key point for its proof is the following property of a circumference: given $m+1$ points $P_1,\ldots,P_{m+1} \in \partial U$ with $m \geqslant 3$ satisfying

$$P_1,\ldots,P_m \text{ are distinct, } P_1 = P_{m+1},\ |P_i - P_{i+1}| < \delta\,\forall\, i = 1,\ldots,m\,,$$

then

$$\sum_{i=1}^m |P_i - P_{i+1}| \geqslant \delta\,.$$

Therefore, by (3.7), since

$$\delta = \mathcal{H}^1_\delta(\partial U) \leqslant \mathcal{H}^1_\delta(\bar{U}) \leqslant \delta\,,$$

it follows that

$$\mathcal{H}^1_\delta(\partial U) = \mathcal{H}^1_\delta(\bar{U}) = \delta\,. \tag{3.8}$$

Let $\sigma \in (0,\delta)$ and notice that, by (3.8) and the subadditivity of $\mathcal{H}^1_\sigma$

$$\sigma = \mathcal{H}^1_\sigma(\bar{U}_\sigma) \leqslant \mathcal{H}^1_\sigma(U)\,.$$

Prove now that

$$\delta \leqslant \liminf_{\sigma\to\delta^-} \mathcal{H}^1_\sigma(U) \leqslant \mathcal{H}^1_\delta(U) \leqslant \delta\,. \tag{3.9}$$

By (3.9) and (3.8), the conclusion of claim (ii) follows if $n = 2$ and $s = 1$.

2nd step: Let

$$n \geqslant 3,\ s = 1 \text{ and } \Gamma := \{x = (x_1,\ldots,x_n) \in \partial U : x_3 = \cdots = x_n = 0\}\,.$$

Then it follows that

$$\mathcal{H}^1_\delta(\Gamma) \leqslant \mathcal{H}^1_\delta(\partial U) \leqslant \delta\,.$$

On the other hand, since Γ is isometric to the circle ∂U of $\mathbb{R}^2$, arguing as in the first step, it follows that

$$\mathcal{H}^1_\delta(\Gamma) = \delta\,.$$

Therefore it follows that

$$\mathcal{H}^1_\delta(\partial U) = \delta\,.$$

Analogously, arguing as in the remaining cases of the first step, the conclusion of claim (ii) follows if $n \geqslant 2$ and $s = 1$.

3rd step: Let

$$n \geqslant 2,\ 0 < s < 1 .$$

It is trivial that

$$\mathcal{H}^s_\delta(\partial U) \leqslant \mathcal{H}^s_\delta(\bar{U}) \leqslant \alpha_s\,\delta^s \text{ and } \mathcal{H}^s_\delta(U) \leqslant \alpha_s\,\delta^s .$$

Observe now that, for each non negative sequence of real numbers $(a_i)_i$,

$$\left(\sum_{i=1}^\infty a_i\right)^s \leqslant \sum_{i=1}^\infty a_i^s .$$

By the 2nd step, this implies that

$$\delta^s = \big(\mathcal{H}^1_\delta(\partial U)\big)^s \leqslant \alpha_s^{-1}\,\mathcal{H}^s_\delta(\partial U) \leqslant \alpha_s^{-1}\mathcal{H}^s_\delta(\bar{U}) \leqslant \delta^s ,$$

and

$$\delta^s = \big(\mathcal{H}^1_\delta(U)\big)^s \leqslant \alpha_s^{-1}\,\mathcal{H}^s_\delta(U) \leqslant \delta^s .$$

Thus the conclusion of claim (ii) follows if $n \geqslant 2$ and $0 < s < 1$.)

Remark 3.4. Observe that pre-measure $\mathcal{H}^s_\delta$ does not fit our project to define a measure which gives back the usual measure on submanifold of $\mathbb{R}^n$. Indeed, let $X = \mathbb{R}^2$, $s = 1$ and $\Gamma := \{(x, x\sin(1/x)) : 0 < x \leqslant 1/\pi\}$. Then it is well-known that length$(\Gamma) = \infty$, but

$$\mathcal{H}^1_\delta(\Gamma) < \infty \quad \forall\, 0 < \delta < \infty .$$

Remark 3.5. We can consider in the previous procedure, in place of $\mathcal{F} = \mathcal{P}(X)$, the family $\mathcal{F}$ of all closed balls of (X, d) and still the evaluation function in (3.4). As before, we can define pre-measure $\mathcal{S}^s_\delta := \psi_\delta$ (see (3.1)) and the resulting measure $\psi := \lim_{\delta\to 0}\psi_\delta$ (see (3.2)) is the so-called *s-dimensional spherical Hausdorff* denoted by $\mathcal{S}^s$. Measures $\mathcal{H}^s$ and $\mathcal{S}^s$ can differ, but they are equivalent. Indeed it holds

$$\mathcal{H}^s(A) \leqslant \mathcal{S}^s(A) \leqslant 2^s\,\mathcal{H}^s(A) \quad \forall\, A \subset X . \tag{3.10}$$

The integral dimensional Hausdorff measures play a special role. Let us start from $s = 0$. It is easy to see that $\mathcal{H}^0$ agrees with the counting measure # on X defined in Example 1.2 (i) Next, for $s = 1$, $\mathcal{H}^1$ also has a concrete interpretation as a generalized length measure. In particular, for a rectifiable curve Γ in $\mathbb{R}^n$, $\mathcal{H}^1(\Gamma)$ can be shown to equal the length of Γ (see Theorem 3.25). For unrectifiable curves $\mathcal{H}^1(\Gamma) = \infty$.

More generally, if m is an integer, $1 < m < n$, and S is a sufficiently regular m-dimensional surface in $\mathbb{R}^n$ (for example, a $\mathbf{C}^1$ submanifold), then the restriction $\mathcal{H}^m \llcorner S$ gives the surface measure on S, for a suitable choice of α_m, as we will see. This follows for example from the area formula, of which we will deal with in the next section.

For $s = n$ in $\mathbb{R}^n$,

$$\mathcal{L}^n = \mathcal{H}^n , \tag{3.11}$$

by choosing the constant $\alpha_n := \dfrac{\mathcal{L}^n(B(1))}{2^n}$ in the definition of $\mathcal{H}^n$. Observe that, by (3.11), it follows that

$$\mathcal{H}^n(B(x,r)) = 2^n\,\alpha_n\,r^n \quad \forall\, x \in \mathbb{R}^n,\ r > 0\,. \tag{3.12}$$

The proof of the equality (3.11) is rather complicated and based on the so-called *isodiametric inequality*

$$\mathcal{L}^n(A) \leqslant \alpha_n\, d(A)^n \quad \text{for } A \subset \mathbb{R}^n$$

which we will deal with in the next section. But to see that $\mathcal{H}^n = c\,\mathcal{L}^n$ with some positive and finite constant c is much easier and a proof is given in [Ma, 4.3] and we will give an other proof in Theorem 3.30.

For any $s > n$, $\mathcal{H}^s$ turns out to be a trivial null measure on $\mathbb{R}^n$. Indeed we will show that $\mathcal{H}^s(\mathbb{R}^n) = 0$ (see Theorem 3.11).

We shall now derive some simple properties of Hausdorff measures in a general separable metric space (X, d).

Theorem 3.6. *Let $s \in [0,\infty)$, $\alpha_s > 0$ and $\zeta(E) := \alpha_s\, d(E)^s$ for $E \subset X$. If*

(i) $\mathcal{F} = \{F \subset X : F$ *closed* $\}$

or

(ii) $\mathcal{F} = \{U \subset X : F$ *open* $\}$,

then $\psi(\mathcal{F},\zeta) = \mathcal{H}^s$, where $\psi(\mathcal{F},\zeta)$ is the set function defined in (3.2).

Proof. Assume that (i) holds. It is clear that, by definition,

$$\mathcal{H}^s \leqslant \psi(\mathcal{F},\zeta)\,. \tag{3.13}$$

In order to prove the reverse inequality, let us first observe that, given $E \subset X$, then $E \subset \bar{E}$ and $d(\bar{E}) = d(E)$ if $\bar{E}$ denotes the closure of E. Assume that $\mathcal{H}^s(A) < \infty$, otherwise we are done. Then, by definition, $\mathcal{H}^s_\delta(A) < \infty$, for each $\delta \in (0,\infty]$. Therefore, for fixed $\delta \in (0,\infty]$ and each $\epsilon > 0$, there is a sequence of sets $(E_i)_i$ such that that $A \subset \cup_{i=1}^\infty E_i \subset \cup_{i=1}^\infty \bar{E}_i$ with $d(E_i) = d(\bar{E}_i) < \delta$, and

$$\psi_\delta(\mathcal{F},\zeta)(A) \leqslant \alpha_s \sum_{i=1}^\infty d(\bar{E}_i)^s = \alpha_s \sum_{i=1}^\infty d(E_i)^s \leqslant \mathcal{H}^s_\delta(A) + \epsilon\,.$$

Passing to the limit as $\delta \to 0$ in the previous inequality and since ϵ is arbitrary, we get the desired inequality. Assume now that (ii) holds. Let us first observe that, for each $E \subset X$ and $\sigma > 0$, $I_\sigma(E) := \{x \in X : d(x,E) < \sigma\}$ is an open set with

$$I_\sigma(E) \supset E \text{ and } d(I_\sigma(E)) \leqslant d(E) + 2\sigma\,.$$

Again, (3.13) is immediate. Let us prove the reverse inequailty. As before, we can assume that, for fixed $\delta \in (0,\infty]$ and each $\epsilon > 0$, there is a sequence of sets $(E_i)_i$ such that that $A \subset \cup_{i=1}^\infty E_i$ with $d(E_i) < \delta$, and

$$\alpha_s \sum_{i=1}^\infty d(E_i)^s \leqslant \mathcal{H}^s_\delta(A) + \epsilon\,. \tag{3.14}$$

For given $i \in \mathbb{N}$ and $\epsilon > 0$, by the continuity of function $[0, \infty) \ni \sigma \mapsto (d(E_i) + 2\sigma)^s$, there exists $\sigma_i = \sigma(i, s, \epsilon) > 0$ such that

$$(d(E_i) + 2\sigma_i)^s < d(E_i)^s + \frac{\epsilon}{\alpha_s\, 2^i}\,. \tag{3.15}$$

Let $U_i := I_{\sigma_i}(E)$ for $i \in \mathbb{N}$. Therefore, by (3.14) and (3.15), it follows that

$$\begin{aligned}\psi_\delta(\mathcal{F}, \zeta)(A) &\leqslant \alpha_s \sum_{i=1}^{\infty} d(U_i)^s \leqslant \alpha_s \sum_{i=1}^{\infty} (d(E_i) + 2\sigma_i)^s \\ &\leqslant \alpha_s \sum_{i=1}^{\infty} d(E_i)^s + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \leqslant \mathcal{H}^s_\delta(A) + 2\epsilon\,.\end{aligned}$$

Passing to the limit as $\delta \to 0$ in the previous inequality and since ϵ is arbitrary, we get the desired inequality. □

Combining Theorems 3.6 and 3.3, we get

Corollary 3.7. *$\mathcal{H}^s$ is a Borel regular outer measure.*

Notice that usually $\mathcal{H}^s$ is not a Radon outer measure since it need not be locally finite. For example, if $s < n$ every non-empty open set in $\mathbb{R}^n$ has non σ-finite $\mathcal{H}^s$ measure. But taking any $\mathcal{H}^s$-measurable set A in $\mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$, the restriction $\mathcal{H}^s \llcorner A$ is a Radon measure by Theorem 1.92 (ii).

Often one is only interested in knowing which sets have null $\mathcal{H}^s$-measure. For this it is enough to use any of the pre-measures $\mathcal{H}^s_\delta$, for example $\mathcal{H}^s_\infty$. In fact we do not need any measure for defining the null $\mathcal{H}^s$-measure sets.

Lemma 3.8 ($\mathcal{H}^s$-null sets)**.** *Let $A \subset X$, $0 < s < \infty$ and $0 < \delta \leqslant \infty$. Then the following conditions are equivalent:*

(i) $\mathcal{H}^s(A) = 0$.
(ii) $\mathcal{H}^s_\delta(A) = 0$.
(iii) $\forall\, \epsilon > 0\ \exists\, E_1, E_2, \dots \subset X$ *such that*

$$A \subset \cup_{i=1}^{\infty} E_i \text{ and } \sum_{i=1}^{\infty} d(E_i)^s < \epsilon\,.$$

Proof. Implications (i)⇒ (ii) ⇒ (iii) are trivial. Let us prove implication (iii) ⇒ (i). Without loss of generality we can assume that $s > 0$, otherwise the conclusion is trivial. By assumptions, it follows that

$$d(E_i) \leqslant \delta(\epsilon) := \epsilon^{1/s} \quad \forall\, i \in \mathbb{N}\,.$$

Therefore

$$\mathcal{H}^s_{\delta(\epsilon)}(A) \leqslant \epsilon \quad \forall\, \epsilon > 0.$$

Passing to the limit as $\epsilon \to 0$ in the previous inequality, we get the desired conclusion. □

We will now compare measures $\mathcal{H}^s$ among them.

Theorem 3.9. *For $0 < s < t < \infty$ and $A \subset X$,*

(i) $\mathcal{H}^s(A) < \infty$ *implies* $\mathcal{H}^t(A) = 0$,
(ii) $\mathcal{H}^t(A) > 0$ *implies* $\mathcal{H}^s(A) = \infty$.

Proof. (i) Let $A \subset \cup_{i=1}^{\infty} E_i$ with $d(E_i) \leqslant \delta$ and $\alpha_s \sum_{i=1}^{\infty} d(E_i)^s < \mathcal{H}^s_\delta(A) + 1 < \infty$. Then

$$\begin{aligned}\mathcal{H}^t_\delta(A) &< \alpha_t \sum_{i=1}^{\infty} d(E_i)^t = \alpha_t \sum_{i=1}^{\infty} d(E_i)^{t-s} d(E_i)^s \\ &< \alpha_t \, \delta^{t-s} \sum_{i=1}^{\infty} d(E_i)^s \leqslant \frac{\alpha_t}{\alpha_s} \, \delta^{t-s} \left(\mathcal{H}^s_\delta(A) + 1\right),\end{aligned}$$

which gives (i) as $\delta \to 0$.

(ii) This claim is only a restatement of (i). □

Hausdorff dimension

Theorem 3.9 enables the definition of an important notion in GMT, namely the Hausdorff dimension. Very roughly, according to [Fa, Introduction], " a dimension provides a description of how much space a set fills. It is a measure of the prominence of the irregularities of a set when viewed at very small scales. A dimension contains much information about the geometrical properties of a set."

Definition 3.10. Let (X, d) be a separable metric space, the *Hausdorff* (or also *metric*) *dimension of a set* $A \subset X$

$$\begin{aligned}\mathrm{Hdim}(A) &= \sup\{s : \mathcal{H}^s(A) > 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\} \\ &= \inf\left\{t : \mathcal{H}^t(A) < \infty\right\} = \inf\left\{t : \mathcal{H}^t(A) = 0\right\}\end{aligned}$$

where we put $\sup \emptyset = 0$ and $\inf \emptyset = \infty$ whether someone of the previous sets may be empty.

The Hausdorff dimension has the natural properties of monotonicity and stability with respect to countable unions:

$$\mathrm{Hdim}(A) \leqslant \mathrm{Hdim}(B) \quad \text{for } A \subset B \subset X\,, \tag{3.16}$$

$$\mathrm{Hdim}(\cup_{i=1}^{\infty} A_i) = \sup_i \mathrm{Hdim}(A_i) \quad \text{for } A_i \subset X,\ i = 1, 2, \dots \tag{3.17}$$

To state the definition in other words, $\mathrm{Hdim}(A)$ is the unique number (it may be ∞ in some metric spaces) for which

$$s < \mathrm{Hdim}(A) \Rightarrow \mathcal{H}^s(A) = \infty,\ t > \mathrm{Hdim}(A) \Rightarrow \mathcal{H}^t(A) = 0\,. \tag{3.18}$$

Remark 3.11. At the borderline case $s = \mathrm{Hdim}(A)$ we cannot have any general nontrivial information about the value $\mathcal{H}^s(A)$; all three cases $\mathcal{H}^s(A) = 0$, $0 < \mathcal{H}^s(A) < \infty$, $\mathcal{H}^s(A) = \infty$. If for some given A there is a $s \geqslant 0$ such that $0 < \mathcal{H}^s(A) < \infty$, then s must equal $\mathrm{Hdim}(A)$.

Remark 3.12. Since, we will see in the next section that $\mathrm{Hdim}(\mathbb{R}^n) = n$ (see Corollary 3.29). Hence $0 \leqslant \mathrm{Hdim}(A) \leqslant n$ for all $A \subset \mathbb{R}^n$. One can prove that, for all $s \in [0, n]$, $\mathrm{Hdim}(A) = s$ for some subset A of $\mathbb{R}^n$ (see Remark 3.35).

3.2. **Recalls of some fundamental results on Lipschitz functions between Euclidean spaces and relationships with Hausdorff measures.**

Recalls of some fundamental results on Lipschitz functions between Euclidean spaces.

Let us recall the general definition of Lipschitz function, which makes sense even for functions acting between metric spaces.

Definition 3.13. Let (X, d) and (Y, ϱ) be metric spaces, let $E \subset X$ and let $f : E \subset X \to Y$.

(i) f is said to be Lipschitz or *L-Lipschitz* if there is $L \geqslant 0$

$$\varrho(f(x_1), f(x_2)) \leqslant L\, d(x_1, x_2) \quad \forall\, x_1, x_2 \in E\,. \tag{3.19}$$

The smallest constant L such that (3.19) holds is called *Lipschitz constant of* f and denoted

$$Lip(f, E) := \sup\left\{\frac{\varrho(f(x_1), f(x_2))}{d(x_1, x_2)} : x_1,\, x_2 \in E,\, x_1 \neq x_2\right\} \in [0, \infty)\,.$$

(ii) f is said to be *locally Lipschitz* if for any compact subset of E there is $L > 0$ such that (3.19) holds for all x_1, x_2 in the compact.

An important issue in GMT and, more generally, in Analysis is the extension of a C^1 or Lipschitz function $f : E \subset \mathbb{R}^n \to \mathbb{R}$ to the whole $\mathbb{R}^n$. Let us first recall a fundamental result due to Whitney about C^1 function.

Theorem 3.14 (Whitney's extension theorem)**.** *Let C be a closed set in $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ and $v : C \to \mathbb{R}^n$ be continuous functions. Define*

$$R(x, y) := \frac{f(x) - f(y) - v(y) \cdot (x - y)}{|x - y|} \quad \forall\, x, y \in C, x \neq y\,.$$

Suppose that for all compact sets $K \subset C$

$$\lim_{r \to 0} \sup\{|R(x, y)| : x,\, y \in K, 0 < |x - y| < r\} = 0\,. \tag{3.20}$$

Then there is $\hat{f} \in \mathbf{C}^1(\mathbb{R}^n)$ such that $\hat{f}|_C = f$ and $\nabla \hat{f}|_C = v$.

Proof. The proof can be found in [EG, Sect. 6.5]. □

Remark 3.15. Let us observe that, by classical Taylor's formula, condition (3.20) is satisfied if $f \in \mathbf{C}^1(\mathbb{R}^n)$. Thus Whiteney's extension theorem is actually a characterization of the $\mathbf{C}^1$-extension of a function defined on a closed set, and is a partial converse to Taylor's formula.

A **real valued** L-Lipschitz function, defined on a subset E of a metric space (X, d), can always be extended to a L-Lipschitz function defined on the entire space X.

Theorem 3.16 (Mc Shane's extension theorem [McS])**.** *Let (X, d) be a metric space and $f : E \subset X \to \mathbb{R}$ be L-Lipschitz. Then there is $\hat{f} : X \to \mathbb{R}$ such that $\hat{f}|_E = f$ and $\hat{f}$ is L-Lipschitz.*

Proof. The proof is not difficult and can be found in the original Mc Shane's paper [McS] as well as in many textbooks devoted to analysis in metric spaces (see, for instance, [AT, Theorem 3.1.2] and [He, He2]). □

Let us recall that as an immediate consequence of Mc Shane's extension theorem is the following extension theorem for Lipschitz maps with target an Euclidean space.

Corollary 3.17. *Let $f : E \to \mathbb{R}^m$, $E \subset (X, d)$ be an L-Lipschitz function. Then there exists an $\sqrt{m}L$-Lipschitz function $\hat{f} : X \to \mathbb{R}^m$ such that $\hat{f}|_E = f$.*

Corollary 3.17 follows by applying Theorem 3.16 to the coordinate functions of f. The multiplicative constant $\sqrt{m}$ in the corollary is in fact redundant, but this is harder to prove.

Theorem 3.18 (Kirszbraun's theorem)**.** *Let $f : E \to \mathbb{R}^m$, $E \subset \mathbb{R}^n$, be an L-Lipschitz function. Then there exists an L-Lipschitz function $\hat{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\hat{f}|_E = f$.*

Proof. See, for instance, [He2, Theorem 2.5]. □

An other fundamental result concerning Lipschitz functions, acting between Euclidean spaces, concerns their differentiability a.e.

Theorem 3.19 (Rademacher's theorem[Rad])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then f is differentiable (in classical sense) $\mathcal{L}^n$-a.e., that is,*

$$\exists\, \nabla f(x) := (\partial_1 f(x), \dots, \partial_n f(x)) \;\; \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n$$

and

$$\lim_{y \to x} \frac{f(y) - f(x) - df(x)(y - x)}{|y - x|} = 0 \tag{3.21}$$

where $df(x) : \mathbb{R}^n \to \mathbb{R}$ denotes the (linear) differential map of f at x defined by

$$df(x)(v) := \nabla f(x) \cdot v \quad \forall\, v \in \mathbb{R}^n\,.$$

Moreover $\nabla f \in (L^\infty_{\text{loc}}(\mathbb{R}^n))^n$.

Proof. See, for instance, [EG, Theorem 2, Sect. 3.1.2]. □

Remark 3.20. Let us point out that, in the 1-dimensional case, i.e. $n = 1$, Rademacher's theorem is an immediate consequence of the fundamental theorem of calculus, since a locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is locally absolutely continuous, i.e. $f|_{[a,b]} \in AC([a, b])$ for each a, $b \in \mathbb{R}$ with $a < b$ (see, for instance, [SC]).

Rademacher's theorem trivially extends to locally Lipschitz functions $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. In this case we get the existence, $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, of the *Jacobian matrix of f* at x , denoted $Df(x)$ and defined by

$$Df(x) := \begin{bmatrix} \partial_1 f_1(x) & \dots & \partial_n f_1(x) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \partial_1 f_m(x) & \dots & \partial_n f_m(x) \end{bmatrix}_{m \times n} . \tag{3.22}$$

Moreover (3.21) now holds with the (linear) differential map $df(x) : \mathbb{R}^n \to \mathbb{R}^m$

$$df(x)(v) := Df(x) \cdot v \quad \forall\, v \in \mathbb{R}^n\,,$$

where the previous product has to be meant as product between $m \times n$ matrix $Df(x)$ and the (column) vector v.

Whitney's theorem together with Rademacher's theorem yield an approximation of a Lipschitz function, which states that it coincides, up to a small set, with a $\mathbf{C}^1$ function.

Theorem 3.21 (Approximation of Lipschitz functions)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then for each $\epsilon > 0$ there is a $g \in \mathbf{C}^1(\mathbb{R}^n)$ such that*

$$\mathcal{L}^n\left(\{x : f(x) \neq g(x)\} \cup \{x : \nabla f(x) \neq \nabla g(x)\}\right) < \epsilon\,.$$

In addition, there is a positive constant $c = c(n)$ such that

$$\sup_{\mathbb{R}^n} |\nabla g| \leqslant c\, Lip(f)\,.$$

Proof. See [EG, Sect. 6.6.1]. □

Let us now stress this simple relationship between Hausdorff measures and Lipschitz maps.

Theorem 3.22 (Hausdorff measures vs. Lipschitz maps)**.** *Let $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. Then*

$$\mathcal{H}^s(f(E)) \leqslant Lip(f)^s\, \mathcal{H}^s(E) \quad \forall\, 0 \leqslant\ s <\ \infty\,.$$

In particular

$$\mathrm{Hdim}(f(E)) \leqslant\ \mathrm{Hdim}(E)$$

Proof. Let $(E_i)_i$ be a countable covering of E by sets with diameter less than δ. Then $(f(E_i))_i$ is a covering of $f(E)$ with

$$\mathrm{diam}(f(F_i)) \leqslant\ Lip(f)\mathrm{diam}(F_i) \leqslant\ Lip(f)\delta \quad \forall\, i\,.$$

Exploiting the arbitrariness of $(E_i)_i$ in the following inequalities,

$$\mathcal{H}^s_{Lip(f)\delta}(f(E)) \leqslant\ \alpha_s \sum_{i=1}^{\infty} \mathrm{diam}(f(E_i))^s \leqslant\ Lip(f)^s\, \alpha_s \sum_{i=1}^{\infty} \mathrm{diam}(E_i)^s\,,$$

we get

$$\mathcal{H}^s_{Lip(f)\delta}(f(E)) \leqslant Lip(f)^s\, \mathcal{H}^s(E)\,.$$

We let $\delta \to 0^+$ to get the desired inequality.

□

Remark 3.23. By Theorem 3.22, we find that Hausdorff measures are decreased under projection over an affine subspace of $\mathbb{R}^n$. Indeed, if H is an affine subspace of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the projection of $\mathbb{R}^n$ over H, then $Lip(f) = 1$. The same happens, of course, if we project over a convex set.

Remark 3.24. We say that $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ $(1 \leqslant\ n \leqslant\ m\ )$ is an *isometry* if $|f(x) - f(y)| = |x - y|$ for every $x,\, y \in E$.

If $s \geqslant\ 0$, $E \subset \mathbb{R}^n$, and $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ is an isometry, then $\mathcal{H}^s(f(E)) =\ \mathcal{H}^s(E)$, as we may see either by applying Theorem 3.22 to f and to any extension g of f^{-1} with $Lip(g) \leqslant\ 1$, or by the area formula (IAF). In particular, if π is an n-dimensional plane in $\mathbb{R}^m$, then there exists an orthogonal injection $P : \mathbb{R}^n \to \mathbb{R}^m$, such that $\pi = P(\mathbb{R}^n)$, that is, there exists $P : \mathbb{R}^n \to \mathbb{R}^m$ injective and satisfying

$$(P(x), P(y))_{\mathbb{R}^m} =\ (x, y)_{\mathbb{R}^n} \quad \forall\, x,\, y \in \mathbb{R}^n \text{ and } P(\mathbb{R}^n) =\ \pi\,.$$

Indeed, let $v_1, \ldots, v_n \in \pi$ be a orthonormal basis of π with respect to the scalar product of $\mathbb{R}^m$. Then

$$P(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i v_i \text{ if } (x_1, \ldots, x_n) \in \mathbb{R}^n$$

turns out to be the desired function. In particlar, notice that P is an isometry It follows that

$$\mathcal{H}^n \llcorner \pi = P_\# \mathcal{H}^n .$$

On the left-hand side, $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure on $\mathbb{R}^m$, on the right-hand side, it denotes the n-dimensional Hausdorff measure on $\mathbb{R}^n$ (which in turn coincides with $\mathcal{L}^n$; see Theorem 3.31). Indeed, notice that, if $A \subset \mathbb{R}^m$,

$$\pi \cap A = P\left(\mathbb{R}^n \cap P^{-1}(A)\right) .$$

Since $P : \mathbb{R}^n \to \mathbb{R}^m$ is an isometry, we get

$$\begin{aligned} \mathcal{H}^n(\pi \cap A) &= \mathcal{H}^n\left(P\left(\mathbb{R}^n \cap P^{-1}(A)\right)\right) \\ &= \mathcal{H}^n\left(\mathbb{R}^n \cap P^{-1}(A)\right) = P_\# \mathcal{H}^n(A) . \end{aligned}$$

Thus we get the desired identity.

3.3. **Hausdorff measures in the Euclidean spaces; $\mathcal{H}^1$ and the classical notion of length in $\mathbb{R}^n$; isodiametric inequality and identity $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.**
In the following of the section, we will assume that

$$X = \mathbb{R}^n \text{ equipped with the Euclidean distance},$$

and we will fix constant α_s in (3.6) in such a way

$$\alpha_s^* := 2^s \, \alpha_s := \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2} + 1\right)} \tag{3.23}$$

where $\Gamma(t) := \int_0^\infty e^{-t} \, x^{t-1} \, dx$ if $t \geqslant 0$ is the Euler Gamma function.

Observe that, if s is equal to a positive integer n then α_n^* agrees with the n-dimensional Lebesgue measure of a unit ball in $\mathbb{R}^n$.

We are going to study here the further properties of Hausdorff measures taking the structure of $\mathbb{R}^n$ into account.

First, Hausdorff measures behave nicely under translations and dilations in $\mathbb{R}^n$: for $A \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$, $0 < t < \infty$,

$$\mathcal{H}^s(A + a) = \mathcal{H}^s(A) \text{ where } A + a := \{x + a : x \in A\}, \tag{3.24}$$

$$\mathcal{H}^s(tA) = t^s \mathcal{H}^s(A) \text{ where } t\, A := \{tx : x \in A\} . \tag{3.25}$$

These are readily verified from the definition.

Hausdorff measures and length measure

A *curve* of $\mathbb{R}^n$ is a continuous function $\gamma : [a, b] \to \mathbb{R}^n$; its *support* is the set $\gamma([a, b]) = \Gamma \subset \mathbb{R}^n$ and, in this case, γ is called a *parametrization* of Γ. For the sake of simplicity, we can assume that $a = 0$ and $b = a$. Given a curve $\gamma : [0, a] \to \mathbb{R}^n$

and a subinterval $[c,d] \subseteq [0,a]$, we define the *length* (or also *variation*) *of* γ *over* $[c,d]$ as
(3.26)

$$l(\gamma;[c,d]) := \sup\left\{\sum_{i=1}^{m} |\gamma(t_i)-\gamma(t_{i-1})| : \ t_0 = c < t_1 < \cdots < t_m = d\right\} \in [0,\infty]$$

where the supremum is taken over all finite partitions $\{t_0 = c < t_1 < \cdots < t_m = d\}$ of $[c,d]$. Moreoever let us denote

$$l(\gamma) := l(\gamma;[a,b]) .$$

Exercise:

(i) $l(\gamma;[b,c]) \geqslant |\gamma(b)-\gamma(c)|$, whenever $0 \leqslant b \leqslant c \leqslant a$;
(ii) $l(\gamma;[b,c]) = l(\gamma;[b,d]) + l(\gamma;[d,c])$, whenever $0 \leqslant b \leqslant d \leqslant c \leqslant a$.

It is also well-known that, if $\gamma : [0,a] \to \mathbb{R}^n$ is of class $\mathbf{C}^1$, then

$$l(\gamma;[c,d]) = \int_c^d |\gamma'(t)|\, dt \quad \forall [c,d] \subset [0,a] . \tag{3.27}$$

If $l(\gamma;[0,a]) < \infty$, the curve γ is said to be *rectifiable.* Whether $l(\gamma;[0,a])$ is finite or not, the following theorem holds true.

Theorem 3.25 (Classical length and $\mathcal{H}^1$)**.** *Let* $\gamma : [0,a] \to \mathbb{R}^n$ *be a curve and denote* $\Gamma = \gamma([0,a])$ *its support. Then*

$$\mathcal{H}^1(\Gamma) \leqslant l(\gamma)$$

and equality holds if $\gamma : [0,a] \subset \mathbb{R} \to \mathbb{R}^n$ *is injective.*

Before the proof we need the following preliminary result and thanks to G.P. Leonardi for usefuel suggestions for the proof.

Lemma 3.26. *Let* $\gamma : [0,a] \to \mathbb{R}^n$ *be a rectifiable curve, that is* $l := l(\gamma;[0,a]) < \infty$*. Let* $v : [0,a] \to [0,l]$ *be the function defined by*

$$v(t) := l(\gamma;[0,t]) \quad t \in [0,a] . \tag{3.28}$$

Then v *is a non decreasing continuous function. In particular* $v([0,a]) = [0,l]$.

Proof. The feature that v is non decreasing is straightforward. Let us prove that v is continuous. Since v is non decreasing, we have only to prove that

$$\lim_{s\to t^+} v(s) = \lim_{s\to t^-} v(s) = v(t) \quad \forall\, t \in [0,a] .$$

Let us prove that

$$\lim_{s\to t^-} v(s) = v(t) \quad \forall\, t \in (0,a] . \tag{3.29}$$

Since $l(\gamma;[0,a]) < \infty$, for each $\epsilon > 0$ there exists a partition of $[0,a]$, $t_0 = 0 < t_1 < \cdots < t_m = a$, such that

$$l(\gamma;[0,a]) - \sum_{i=0}^{m} |\gamma(t_i)-\gamma(t_{i-1})| < \epsilon . \tag{3.30}$$

Moreover, without loss of generality, we can suppose that $t_{i_0} = t$ for some $i_0 = 1, \ldots, m$. Otherwise, since there exists $i_0 = 1, \ldots, m$ such that $t_{i_0 - 1} < t < t_{i_0}$, we could enlarge the previuos partition by defining a new partition

$$t_i^* := \begin{cases} t_i & \text{if } 0 \leqslant i \leqslant i_0 - 1 \\ t & \text{if } i = i_0 \\ t_{i+1} & \text{if } i_0 \leqslant i \leqslant m \end{cases}$$

and, because of

$$\sum_{i=0}^{m} |\gamma(t_i) - \gamma(t_{i-1})| \leqslant \sum_{i=0}^{m+1} |\gamma(t_i^*) - \gamma(t_{i-1}^*)|\,,$$

we still have

$$\begin{aligned} &l(\gamma;[0,a]) - \sum_{i=0}^{m+1} |\gamma(t_i^*) - \gamma(t_{i-1}^*)| \\ &\leqslant l(\gamma;[0,a]) - \sum_{i=0}^{m} |\gamma(t_i) - \gamma(t_{i-1})| < \epsilon\,. \end{aligned}$$

Observe now, by claim (ii) of the previous exercise, we can write (3.30) as

$$\begin{aligned} &l(\gamma;[0,a]) - \sum_{i=0}^{m} |\gamma(t_i) - \gamma(t_{i-1})| \\ &= \sum_{i=0}^{m} \left(l(\gamma;[t_{i-1}, t_i]) - |\gamma(t_i) - \gamma(t_{i-1})|\right) < \epsilon\,. \end{aligned}$$

which implies,

$$l(\gamma;[t_{i_0-1}, t]) - |\gamma(t) - \gamma(t_{i_0-1})| < \epsilon\,. \tag{3.31}$$

Meanwhile, from (3.31), it follows that, for each $t_{i_0-1} \leqslant s \leqslant t$,

$$\begin{aligned} &l(\gamma;[t_{i_0-1}, t]) - (|\gamma(t_{i_0-1}) - \gamma(s)| + |\gamma(s) - |\gamma(t)|) \\ &\ = l(\gamma;[t_{i_0-1}, s]) + l(\gamma;[s,t]) - (|\gamma(t_{i_0-1}) - \gamma(s)| + |\gamma(s) - |\gamma(t)|) \\ &l(\gamma;[t_{i_0-1}, t]) - |\gamma(t) - \gamma(t_{i_0-1})| < \epsilon\,, \end{aligned}$$

which implies, for each $t_{i_0-1} \leqslant s \leqslant t$,

$$v(t) - v(s) - |\gamma(s) - \gamma(t)| = l(\gamma;[s,t]) - |\gamma(s) - \gamma(t)| < \epsilon\,. \tag{3.32}$$

On the other hand, since γ is continuous, for each $\epsilon > 0$ there is $\bar{t} = \bar{t}(\epsilon) \in (t_{i_0-1}, t)$ such that for each $\bar{t} < s < t$

$$|\gamma(t) - \gamma(s)| < \epsilon\,. \tag{3.33}$$

Therefore, by (3.32) and (3.33), it follows that, for each $\epsilon > 0$ there is $\bar{t} < t$ such that

$$v(t) - v(s) < 2\epsilon \quad \forall\, s \in (\bar{t}, t)\,,$$

and (3.29) follows. To prove the right continuity, that is

$$\lim_{s \to t^+} v(s) = v(t) \quad \forall\, t \in [0, a) \tag{3.34}$$

we can use the same procedure. Indeed, by (3.30), we can now suppose that $t_{i_0-1} = t$ for some $i_0 = 1, \dots, m$. Arguing as before, we get that, for each $t \leqslant s \leqslant t_{i_0}$,

$$v(s) - v(t) - |\gamma(s) - \gamma(t)| = l(\gamma; [t, s]) - |\gamma(s) - \gamma(t)| < \epsilon$$

and then, by the continuity of γ, (3.34) follows. □

Proof of Theorem 3.25. We will follow the proof in [Mag, Theorem 3.8]. Let us begin with the following exercise.

Exercise: Prove that $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$, as outer measures.

(*Hint:* Use the fact that $\mathcal{L}^1(E) \leqslant d(E)$ for each $E \subset \mathbb{R}$.)

The theorem is proved by Remark 3.24 and the previous exercise if Γ is a segment, Indeed, assume that $\Gamma = \gamma([0, a])$ with

$$\gamma(t) := p + t \frac{p - q}{|p - q|} \text{ if } 0 \leqslant t \leqslant a := |p - q|$$

where p, $q \in \mathbb{R}^n$ with $p \neq q$. Since $\gamma : [0, a] \subset \mathbb{R} \to \mathbb{R}^n$ is an isometry, then

$$\begin{aligned} \mathcal{H}^1(\Gamma) &= \mathcal{H}^1 (\gamma([0, a]) = \mathcal{H}^1([0, a]) \\ &= \mathcal{L}^1([0, a]) = |p - q| = l(\gamma; [0, a]) . \end{aligned}$$

Set $l = l(\gamma; [0, a])$. We divide the proof into three steps.

1st step. Let us show that $\mathcal{H}^1(\Gamma) \geqslant |\gamma(a) - \gamma(0)|$. Since the projection $p : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ onto the line defined by $\gamma(0)$ and $\gamma(a)$ satisfies $Lip(p) \leqslant 1$, by Theorem 3.22 we have

$$\mathcal{H}^1(p(\Gamma)) \leqslant \mathcal{H}^1(\Gamma) .$$

At the same time, $p(\Gamma)$ must contain the segment $S := \{t\gamma(a) + (1 - t)\gamma(0) : 0 \leqslant t \leqslant 1\}$. Otherwise, $\Gamma = \gamma([0, a])$ would be disconnected, against the continuity of γ. Thus $\mathcal{H}^1(p(\Gamma)) \geqslant \mathcal{H}^1(S) = |\gamma(a) - \gamma(0)|$.

2nd step. Let us prove that $\mathcal{H}^1(\Gamma) \leqslant l$. If $l = \infty$, we are done. Thus we can assume that $l < \infty$ and we are going to construct a Lipschitz function $\gamma^* : [0, l] \to \mathbb{R}^n$ with $Lip(\gamma^*) \leqslant 1$ and $\Gamma = \gamma^*([0, l])$. Indeed, by Theorem 3.22, the existence of γ^* will imply, as required, that

$$\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\gamma^*([0, l]) \leqslant \mathcal{H}^1([0, l]) = l .$$

First let us assume that γ is injective. To construct γ^* (which is just the parametrization by arc length of γ, defined without using derivatives), we define Then $v(0) = 0$, $v(a) = l$ and v is strictly increasing, that is, $v(t) < v(s)$ if $t < s$, as γ is injective. In particular, v is continuous and invertible, with a continuous strictly increasing inverse $w : [0, l] \to [0, a]$. Let then $\gamma^* : [0, l] \to \mathbb{R}^n$ be defined by

$$\gamma^*(s) := \gamma(w(s)) \quad s \in [0, l] .$$

Then

Exercise: Prove that

$$l(\gamma^*; [0, s]) = s \quad \forall\, s \in [0, l] \tag{3.35}$$

(**Hint:** prove that $l(\gamma^*; [0, s]) = l(\gamma; [0, w(s)]) = v(w(s)) = s$).

We easily find that $Lip(\gamma^*) \leqslant 1$, since, by properties (i) and (ii) in the exercise above, if $[s_1, s_2] \subset [0, l]$, then

$$|\gamma^*(s_1) - \gamma^*(s_2)| \leqslant l(\gamma^*; [s_1, s_2]) = l(\gamma^*; [0, s_2]) - l(\gamma^*; [0, s_2]) = s_2 - s_1 .$$

If γ is not injective, even if the construction is more difficult (see, for instance,[AT, Theorem 4.2.1]), we can still construct a parametrization of Γ, $\gamma^* : [0, l] \to \mathbb{R}^n$ with $Lip(\gamma^*) \leqslant 1$ and we can argue as before.

3rd step. Suppose now that γ is injective.If $t_0 = 0, \dots, t_m = a$ is a competitor in the definition of l, then, setting $\Gamma_h := \gamma([t_{h-1}, t_h])$ $(h = 1, \dots, m)$, we have $\Gamma = \cup_{h=1}^m \Gamma_h$ and, by the injectivity of γ, $\mathcal{H}^1(\Gamma_h \cap \Gamma_{h+1}) = \mathcal{H}^1(\{\gamma(t_h)\} = 0$. We thus find $\mathcal{H}^1(\Gamma) \geqslant l$ as, by step one,

$$\mathcal{H}^1(\Gamma) = \sum_{h=1}^m \mathcal{H}^1(\Gamma_h) \geqslant \sum_{h=1}^m |\gamma(t_h) - \gamma(t_{h-1})| .$$

□

Remark 3.27. When $\gamma : [0, a] \to \mathbb{R}^n$ is of class $\mathbf{C}^1$, it is immediately seen that (3.27) holds with $c = 0$ and $d = a$. In particular, by Theorem 3.25, if γ is injective and $\Gamma = \gamma([0, a])$,

$$\mathcal{H}^1(\Gamma) = \int_0^a |\gamma'(t)| \, dt .$$

This is the one-dimensional case of the area formula discussed in the previous section.

Hausdorff measures and Lebesgue measure

We are going to compare the outer measures $\mathcal{L}^n$ and $\mathcal{H}^s$ on $\mathbb{R}^n$. Let us first estimate the values of $\mathcal{H}^s$ on the balls.

Proposition 3.28.

$$\mathcal{H}^s(B(x, r)) = c(s, n)\, r^s \quad x \in \mathbb{R}^n,\ 0 < r < \infty \tag{3.36}$$

with $c(s, n)$ positive and finite constant only when $s = n$; for $s > n$, $c(s, n) = 0$; for $s < n$, $c(s, n) = \infty$.

Corollary 3.29. (i) *$\mathcal{H}^s$ is a (non trivial) Radon measure on $\mathbb{R}^n$ if and only $s = n$.*

(ii) $\mathrm{Hdim}(A) = n$ *for each (nonemtpy) open set $A \subset \mathbb{R}^n$. In particular* $\mathrm{Hdim}(\mathbb{R}^n) = n$.

Proof. (i) Is is an immeditae consequence of Proposition 3.28.

(ii) By Proposition 3.28 and (3.18), it follows that $\mathrm{Hdim}(U(x, r)) = n$ for each $x \in \mathbb{R}^n$ and $r > 0$, where $U(x, r)$ is an an open ball centered at x and with radius $r > 0$. Indeed, for fixed $r > 0$, let $(r_h)_h$ be a strictly increasing sequence of positive real numbers such that $\lim_{h\to\infty} r_h = r$. Thus, we can write

$$U(x, r) = \cup_{h=1}^\infty B(x, r_h) ,$$

and, by (3.17), we get the desired conclusion. Since by Lemma 1.16, $A = \cup_{i=1}^\infty U_i$ with U_i $(i \in \mathbb{N})$ open balls of $\mathbb{R}^n$, by (3.17) we get the desired conclusion. □

Proof of Proposition 3.28. Let us first observe that, by (3.24) and (3.25), it follows that there exists $c(s,n) := \mathcal{H}^s(B(1)) \in [0,\infty]$ such that (3.36) holds. We have only to show that

$$0 < c(n,n) = \mathcal{H}^n(B(0,1)) < \infty\,. \tag{3.37}$$

Indeed, from Theorem 3.9 and (3.37), it will follow that $c(s,n) = \infty$, if $s < n$ and $c(s,n) = 0$ if $s > n$. Let us observe that, by (3.10), the proof of left-hand side inequality in (3.37) is equivalent to show that

$$\mathcal{S}^n(B(1)) > 0\,, \tag{3.38}$$

where $\mathcal{S}^n$ denotes the n-dimensional spherical Hausdorff measure on $\mathbb{R}^n$. Let us prelinimarly observe that, if $B = B(x,r)$ is a closed ball of $\mathbb{R}^n$ and $Q_B^{(i)}$, $Q_B^{(c)}$ denote, respectively, the inscribed and circumscribed n-dimensional (closed) cube to B, then

$$Q_B^{(i)} \subset B \subset Q_B^{(c)}\,,$$

and their side length are

$$l\left(Q_B^{(i)}\right) = \frac{2r}{\sqrt{n}} \text{ and } l\left(Q_B^{(c)}\right) = 2r\,.$$

In particular the diameters of $Q_B^{(i)}$ and $Q_B^{(c)}$ are

$$d\left(Q_B^{(i)}\right) = 2r = d(B) \text{ and } d\left(Q_B^{(c)}\right) = \sqrt{n}\,2r = \sqrt{n}\,d(B)\,. \tag{3.39}$$

Let us begin to prove (3.38). By definition of the n-dimensional pre-measure spherical Hausdorff measure and (3.39), we get, for each $\delta \in (0,\infty]$,

(3.40)

$$\begin{aligned}
\mathcal{S}^n_\delta(B(1)) &= \inf\left\{\alpha_n \sum_{j=1}^\infty d(B_j)^n : B(1) \subset \cup_{j=1}^\infty B_j,\ B_j \text{ closed ball, } d(B_j) \leqslant \delta\right\} \\
&= \inf\left\{\frac{\alpha_n}{n^{n/2}} \sum_{j=1}^\infty d\left(Q_{B_j}^{(c)}\right)^n : B(1) \subset \cup_{j=1}^\infty B_j,\ B_j \text{ closed ball, } d(B_j) \leqslant \delta\right\} \\
&\geqslant \inf\left\{\alpha_n \sum_{j=1}^\infty \left(\frac{d\left(Q_j\right)}{\sqrt{n}}\right)^n : B(1) \subset \cup_{j=1}^\infty Q_j,\ Q_j \text{ closed cube}\right\} \\
&= \inf\left\{\alpha_n \sum_{j=1}^\infty \mathcal{L}^n(Q_j) : B(1) \subset \cup_{j=1}^\infty Q_j,\ Q_j \text{ closed cube}\right\} \\
&\geqslant \alpha_n\, \mathcal{L}^n(B(1)) > 0\,,
\end{aligned}$$

where the last inequality follows according to one of possible definitions of n- dimensional Lebesgue measure (see, for instance, [GZ, Sect. 4.3]). Passing to the limit as $\delta \to 0$ in (3.40), (3.38) follows. Let $Q_* := Q^{(c)}_{B(1)} = [-1,1]^n$, then the right-hand side inequality in (3.37) will follow if we prove that

$$\mathcal{H}^n(Q_*) < \infty\,. \tag{3.41}$$

For each $h \in \mathbb{N}$, let us divide Q_* in h^n closed cubes Q_k ($k = 1, \dots, h^n$) with side length $2/h$. Then, for each $\delta \in (0, \infty)$, by choosing h be such that

$$d(Q_k) = \frac{\sqrt{n}}{2h} < \delta \quad \forall\, k = 1, \dots, h^n\,,$$

we get

$$\mathcal{H}^n_\delta(Q_*) \leqslant \alpha_n \sum_{k=1}^{h^n} d(Q_k)^n = \frac{\alpha_n\, n^{n/2}}{2^n} \sum_{k=1}^{h^n} \frac{1}{h^n} = \frac{\alpha_n\, n^{n/2}}{2^n} \quad \forall\, \delta \in (0, \infty)\,.$$

By passing to the limit as $\delta \to 0$ in the previous inequality, (3.41) follows. □

By means of Proposition 3.28 and Theorem 2.15, we can infer the agreement, up to a constant, of outer measures $\mathcal{H}^n$ and $\mathcal{L}^n$ on $\mathbb{R}^n$.

Theorem 3.30. *Let* $c := \dfrac{\mathcal{H}^n(B(1))}{\alpha^*_n}$. *Then*

$$\mathcal{H}^n(A) = c\,\mathcal{L}^n(A) \quad \forall\, A \subset \mathbb{R}^n\,. \tag{3.42}$$

Proof. Assume that it holds

$$\mathcal{H}^n(B) = c\,\mathcal{L}^n(B) \quad \forall\, B \in \mathcal{B}(\mathbb{R}^n)\,. \tag{3.43}$$

Let us prove that (3.42) also holds. Indeed, since $\mathcal{H}^n$ and $\mathcal{L}^n$ are regular Borel outer measures, for each $A \subset \mathbb{R}^n$ there exist Borel sets B_i ($i = 1, 2$) such that

$$A \subset B_1 \text{ and } \mathcal{H}^n(B_1) = \mathcal{H}^n(A),\ A \subset B_2 \text{ and } \mathcal{L}^n(B_2) = \mathcal{L}^n(A)\,. \tag{3.44}$$

By (3.43) and (3.44), it follows that

$$\begin{aligned} \mathcal{H}^n(A) &= \mathcal{H}^n(B_1) = c\,\mathcal{L}^n(B_1) \geqslant c\,\mathcal{L}^n(A)\,, \\ c\,\mathcal{L}^n(A) &= c\,\mathcal{L}^n(B_2) = \mathcal{H}^n(B_2) \geqslant \mathcal{H}^n(A)\,. \end{aligned}$$

Thus (3.42) follows. Let us now prove (3.43) . Let $\mu := \mathcal{L}^n$, $\nu := \dfrac{1}{c}\mathcal{H}^n$ and $\lambda := \dfrac{1}{2}(\mu + \nu)$. Then, by Proposition 3.28, μ, ν and λ are positive Radon measures on measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and

$$\mu(B(x, r)) = \nu(B(x, r)) = \lambda(B(x, r)) \quad \forall\, x \in \mathbb{R}^n, r \in (0, \infty)\,. \tag{3.45}$$

Moreover it is trivial that

$$\mu << \lambda \text{ and } \nu << \lambda\,. \tag{3.46}$$

Thus, by (2.19),

$$\mu(B) = \int_B D_\lambda \mu\, d\lambda \text{ and } \nu(B) = \int_B D_\lambda \nu\, d\lambda \quad \forall\, B \in \mathcal{B}(\mathbb{R}^n)\,. \tag{3.47}$$

On the other hand, since

$$D_\lambda \mu(x) = D_\lambda \nu(x) = 1 \quad \forall\, x \in \mathbb{R}^n\,,$$

(3.43) follows. □

We are now going to characterize the constant c in Theorem 3.30. Indeed we will prove that $c = 1$.

Theorem 3.31 ($\mathcal{H}^n \equiv \mathcal{L}^n$)**.** *$\mathcal{L}^n(A) = \mathcal{H}^n_\delta(A) = \mathcal{H}^n(A)$ for each $A \subset \mathbb{R}^n$, $0 < \delta \leqslant \infty$.*

The proof of Theorem 3.31 is based on the isodiametric inequality, which, as said before, is rather complicated. Indeed it follows by Steiner symmetrization, which we will not introduce here. Isodiametric inequality states that Euclidean balls in $\mathbb{R}^n$ are the sets of maximum n-dimensional Lebesgue measure among all sets of a given diameter.

Theorem 3.32 (Isodiametric inequality)**.**

$$\mathcal{L}^n(A) \leqslant \alpha_n\, d(A)^n \quad \text{for } A \subset \mathbb{R}^n\,.$$

Proof. See [EG, Theorem 1, Sect. 2.2]. □

Remark 3.33. Notice that the isodiametric inequality is not trivial: a set of a given diameter is not necessarily contained in a ball of the same diameter.

Exercise: Let A be an equilater triangle of the plane $\mathbb{R}^2$, with side length l. Prove that $d(A) = l$ and there is no closed ball with diameter l containing A.

Proof of Theorem 3.31. **1st step.** Let us first observe that, by Corollary 3.29 (i) , $\mathcal{H}^n$ is a Radon measure on $\mathbb{R}^n$.

2nd step. Let us prove the inequality

$$\mathcal{H}^n_\delta(A) \leqslant \mathcal{L}^n(A) \quad \forall\, A \subset \mathbb{R}^n,\, 0 < \delta \leqslant \infty\,. \tag{3.48}$$

Let $V \subset \mathbb{R}^n$ be an open set such that $A \subset V$ an let

$$\mathcal{G}_\delta := \left\{ B(x,r) :\, x \in A,\, r < \frac{\delta}{2},\, B(x,r) \subset V \right\}\,.$$

By Vitali's covering property for Radon measures (see Theorem 2.11) the exists a disjoint subfamily $\mathcal{F} \subset \mathcal{G}_\delta$, at most countable, such that

$$\mathcal{H}^n(A \setminus \cup\mathcal{F}) = 0\,.$$

By Lemma 3.8, it also follows that

$$\mathcal{H}^n_\delta(A \setminus \cup\mathcal{F}) = 0\,.$$

Therofore, since $\mathcal{H}^n_\delta$ is an outer measure,

$$\begin{aligned} \mathcal{H}^n_\delta(A) &\leqslant \mathcal{H}^n_\delta(\cup\mathcal{F}) \leqslant \sum_{B\in\mathcal{F}} \mathcal{H}^n_\delta(B) \\ &\leqslant \alpha_n \sum_{B\in\mathcal{F}} d(B)^n = \mathcal{L}^n(\cup\mathcal{F}) \leqslant \mathcal{L}^n(V)\,. \end{aligned}$$

Taking the infimum in the previous inequality, over all open sets $V \supset A$, we get (3.48).

3rd step. Let us prove the inequality

$$\mathcal{H}^n_\delta(A) \geqslant \mathcal{L}^n(A) \quad \forall\, A \subset \mathbb{R}^n,\, 0 < \delta \leqslant \infty\,. \tag{3.49}$$

Let $(A_i)_i$ be a sequence of sets such that

$$A \subset \cup_{i=1}^\infty A_i,\; d(A_i) < \delta\; \forall\, i\,. \tag{3.50}$$

Then, by the isodiametric inequality,

$$\mathcal{L}^n(A) \leqslant \sum_{i=1}^{\infty} \mathcal{L}^n(A_i) \leqslant \alpha_n \sum_{i=1}^{\infty} d(A_i)^n .$$

Taking the infimum in the previous inequality, over all sequences $(A_i)_i$ satisfying (3.50), we get (3.49). □

Hausdorff measures and Cantor sets

We have now introduced measures for measuring the size of very general sets. We give a look at some examples with which Hausdorff measures are convenient and useful. We begin with the most classical as the Cantor sets.

Let us first introduce the Cantor sets in $\mathbb{R}$. Let $0 < \lambda < 1/2$ and define by steps the following intervals.

Step $k = 1$**:** denote $I_{0,1} := [0, 1]$ and let us delete the middle open interval of length $(1 - 2\lambda)\, d(I_{0,1})$, that is interval $(\lambda, 1 - \lambda)$. This yields 2 closed intervals

$$I_{1,1} =: [0, \lambda] \text{ and } I_{1,2} =: [1 - \lambda, 1] .$$

Let us define

$$C_1(\lambda) := \cup_{j=1}^{2} I_{1,j} .$$

We continue this process of selecting two subintervals of each already given interval.

Step $k = 2$**:** Let us delete from intervals $I_{1,1}$ and $I_{1,2}$ the open middle intervals of length $(1 - 2\lambda)\, d(I_{1,1}) = (1 - 2\lambda)\, d(I_{1,2}) = (1 - 2\lambda)\, \lambda$. This yields 2^2 closed intervals which can denote from left to right as

$$I_{2,1},\ I_{2,2},\ I_{2,3},\ I_{2,4} .$$

Let us define

$$C_2(\lambda) := \cup_{j=1}^{4} I_{2,j} .$$

Step k**:** if we have defined the 2^{k-1} intervals

$$I_{k-1,1}, \ldots, I_{k-1,2^{k-1}} ,$$

we define 2^k intervals

$$I_{k,1}, \ldots, I_{k,2^k}$$

by deleting from the middle of each $I_{k-1,j}$ an interval of length $(1 - 2\lambda) d(I_{k-1,j}) = (1 - 2\lambda)\lambda^{k-1}$. All the intervals $I_{k,j}$ thus obtained have length

$$d(I_{k,j}) = \lambda^k \quad \forall\, j = 1, \ldots, 2^k .$$

Let us define

$$C_k(\lambda) := \cup_{j=1}^{2^k} I_{k,j} .$$

We define the limit set of this construction by

$$C(\lambda) := \cap_{k=1}^{\infty} C_k(\lambda) . \tag{3.51}$$

Then it is well-known that :

- $C(\lambda)$ is an uncountable compact set ,
- $\overset{\circ}{C}(\lambda) = \emptyset$,
- $\mathcal{H}^1(C(\lambda)) = \mathcal{L}^1(C(\lambda)) = 0$.

If $\lambda = 1/3$, $C(1/3)$ is the celebrated *Cantor middle-third set.*

We shall now study the Hausdorff measures and dimension of $C(\lambda)$. As usual, it is much simpler to find upper bounds than lower bounds for the Hausdorff measures. This is due to the definition: a suitable chosen covering will give an upper estimate, but a lower estimate requires finding an infimum over arbitrary coverings. By using Remark 3.12, our goal is to find out a value $s \in (0,1)$ such that

$$0 < \mathcal{H}^s(C(\lambda)) < \infty \tag{3.52}$$

from which it will follow that $\alpha = \operatorname{Hdim}(C(\lambda))$.

Question: How can we guess a value s satisfying (3.52)?

A possible way is by using the *self-similar structure* of $C(\lambda)$, of which we speak about.

Let $S_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$

$$S_1(x) := \lambda\, x, \quad S_2(x) := \lambda\, x + 1 - \lambda\,.$$

Then S_i are two similarities of $\mathbb{R}$ and

Exercise:

$$S_1\left(C(\lambda)\right) = C(\lambda) \cap [0, \lambda], \quad S_2\left(C(\lambda)\right) = C(\lambda) \cap [1-\lambda, 1]\,.$$

(**Hint:** Let us first prove that $C(\lambda) = S_1(C(\lambda)) \cup S_2(C(\lambda))$, from which the desired conclusion will follow.)

Therefore, by the previous exercise, (3.24) and (3.25), it follows that

$$\begin{aligned}
\mathcal{H}^s\left(C(\lambda)\right) &= \mathcal{H}^s\left(C(\lambda) \cap [0,\lambda]\right) + \mathcal{H}^s\left(C(\lambda) \cap [1-\lambda, 1]\right) \\
&= \mathcal{H}^s\left(S_1(C(\lambda))\right) + \mathcal{H}^s\left(S_2(C(\lambda))\right) \\
&= \lambda^s\, \mathcal{H}^s\left(C(\lambda)\right) + \lambda^s\, \mathcal{H}^s\left(C(\lambda)\right) \\
&= 2\, \lambda^s\, \mathcal{H}^s\left(C(\lambda)\right)\,.
\end{aligned}$$

Thus, if (3.52) holds, then $2\, \lambda^s = 1$, or, equivalently,

$$s = \frac{\log 2}{\log \frac{1}{\lambda}}\,. \tag{3.53}$$

Theorem 3.34 (Hausdorff dimension of the Cantor sets in $\mathbb{R}$)**.** *Let s be the value in* (3.53) *and let α_s be the constant in* (3.23)*. Then*

(i) $\mathcal{H}^s\left(C(\lambda)\right) \leqslant \alpha_s < \infty$*;*

(ii) $\mathcal{H}^s\left(C(\lambda)\right) \geqslant \alpha_s > 0$*,*

In particular $\mathcal{H}^s\left(C(\lambda)\right) = \alpha_s$ and

$$\operatorname{Hdim}(C(\lambda)) = \frac{\log 2}{\log \frac{1}{\lambda}}\,.$$

Remark 3.35. Note that $\operatorname{Hdim}(C(\lambda))$ measures the sizes of the Cantor sets $C(\lambda)$ in a natural way: when λ increases, the sizes of the deleted holes decrease and the sets $C(\lambda)$ become larger, and also $\operatorname{Hdim}(C(\lambda))$ increases. Notice also that when λ runs from 0 to 1/2, $\operatorname{Hdim}(C(\lambda))$ takes all the values between 0 and 1.

Proof. (i) Notice that, by definition,

$$C(\lambda) \subset C_k(\lambda) := \cup_{j=1}^{2^k} I_{k,j} \forall\, k$$

and so, if $\delta = \lambda^k$

$$\mathcal{H}^s_{\lambda^k}(C(\lambda)) \leqslant \alpha_s \sum_{j=1}^{2^k} d(I_{k,j})^s = \alpha_s\, 2^k\, \lambda^{ks} = \alpha_s\, (2\,\lambda^s)^k = \alpha_s\,.$$

Letting $k \to \infty$ in the previous inequality, we get the desired inequality.
(ii) This lower bound is harder and we recommend [Ma, 4.10]. □

Several different Cantor-type sets can be constructed in $\mathbb{R}^n$ still for $n = 1$ that in higher dimensions $n \geqslant 2$ (see [Ma, 4.11-13] and [Fa] for a deeper analysis).

Hausdorff measures as Radon measures

It is immediate that $\mathcal{H}^k \llcorner E$ induces a positive finite measure in $\mathbb{R}^n$ whenever E is $\mathcal{H}^k$ measurable and $\mathcal{H}^k(E) < \infty$. Conversely, in many applications one needs to know whether a given measure μ is representable in terms of the Hausdorff measure, or at least needs to estimate the Hausdorff dimension of the set where μ is concentrated. In order to compare μ with $\mathcal{H}^k$ the natural idea is to look at the ratio $\mu(B(x,r))/\alpha_k^*\, r^k$, and this motivates the following definition.

Definition 3.36 (k-dimensional densities)**.** Let μ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^n$ and $k > 0$. The *upper* and *lower k -dimensional densities of μ at x* are respectively defined by

$$\Theta^*_k(\mu, x) := \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{\alpha_k^*\, r^k}, \quad \Theta_{*k}(\mu, x) := \liminf_{r \to 0^+} \frac{\mu(B(x,r))}{\alpha_k^*\, r^k}$$

If $\Theta^*_k(\mu, x) = \Theta_{*k}(\mu, x)$ their common value is denoted by $\Theta_k(\mu, x)$ and this notation is also used for $\mathbb{R}^m$ -valued Radon measures μ whenever the densities of their component μ_i are defined, i.e. the i-th component of $\Theta_k(\mu, x)$ is $\Theta_k(\mu_i, x)$ for any $i = 1, \ldots, m$. For any Borel set $E \subset \mathbb{R}^n$ we define also

$$\Theta^*_k(E, x) := \limsup_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B(x,r))}{\alpha_k^*\, r^k}, \quad \Theta_{*k}(E, x) := \liminf_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B(x,r))}{\alpha_k^*\, r^k}$$

and, if they agree, we denote the common value of these densities by $\Theta_k(E, x)$.

Clearly $\Theta^*_k(E, x) = \Theta^*_k(\mathcal{H}^k \llcorner E, x)$ and $\Theta_{*k}(E, x) = \Theta_{*k}(\mathcal{H}^k \llcorner E, x)$. Using the left continuity of $(0, \infty) \ni r \mapsto \mu(B(x,r))$ it can be easily checked that all the densities are Borel functions of x. Now we see how the upper density $\Theta^*_k(\mu, x)$ can be used to estimate from below and from above μ with $\mathcal{H}^k$, which will turn out to be very useful in the topic of rectifiable sets.

Theorem 3.37 (Estimates of the upper density of a Radon measure)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and μ a positive Radon measure in Ω. Then, for any $t \in (0, \infty)$ and any Borel set $B \subset \Omega$ the following implications hold:*

$$\Theta^*_k(\mu, x) \geqslant t \quad \forall\, x \in B \quad \Rightarrow \quad \mu \geqslant t\,\mathcal{H}^k \llcorner B\,, \tag{3.54}$$

$$\Theta^*_k(\mu, x) \leqslant t \quad \forall\, x \in B \quad \Rightarrow \quad \mu \leqslant 2^k\, t\,\mathcal{H}^k \llcorner B\,. \tag{3.55}$$

Proof. See, for instance, [AFP, Theorem 2.56]. □

Two fundamental consequences of Theorem 3.37, very useful in the study of measure-theoretic property of sets, are the following.

Corollary 3.38. *Let $k \in [0,n]$ and assume that $E \subset \mathbb{R}^n$ is $\mathcal{H}^k$-measurable and $\mathcal{H}^k(E) < \infty$. Then*

$$\exists\, \Theta_k(E,x) = \lim_{r\to 0^+} \frac{\mathcal{H}^k(E\cap B(x,r))}{\alpha_k^*\, r^k} = 0 \quad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in \mathbb{R}^n \setminus E\,; \tag{3.56}$$

$$2^{-k} \leqslant \limsup_{r\to 0^+} \frac{\mathcal{H}^k(E\cap B(x,r))}{\alpha_k^*\, r^k} \leqslant 1 \quad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in E\,. \tag{3.57}$$

Remark 3.39. If $k = n$, let us recall that, by the result concerning the density of a set (see Corollary 2.18)

$$\lim_{r\to 0^+} \frac{\mathcal{H}^n(E\cap B(x,r))}{\alpha_n^*\, r^n} = \lim_{r\to 0^+} \frac{\mathcal{L}^n(E\cap B(x,r))}{\mathcal{L}^n(B(x,r)} = \chi_E(x) \quad \mathcal{H}^n\text{-a.e. } x \in \mathbb{R}^n\,.$$

It is not the case when $k \in (0,n)$, even though k is integer. Indeed it is possible to have

$$\limsup_{r\to 0^+} \frac{\mathcal{H}^k(E\cap B(x,r))}{\alpha_k^*\, r^k} < 1$$

and

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^k(E\cap B(x,r))}{\alpha_k^*\, r^k} = 0$$

for $\mathcal{H}^k$-a.e. $x \in E$ even if $0 < \mathcal{H}^k(E) < \infty$, for a E not regular in measure-theoretic sense, that is if E is *unrerctifiable* (see Example 4.28).

Proof of Corollary 3.38. (i) Let $t > 0$ and let

$$B_t := \{x \in \mathbb{R}^n \setminus E : \Theta_k^*(E,x) > t\}\,.$$

Then, by (3.54) with $\mu = \mathcal{H}^k \llcorner E$ and $B = B_t$,

$$\begin{aligned} \mathcal{H}^k(B_t) &= (\mathcal{H}^k \llcorner B_t)(B_t) \leqslant \frac{1}{t}\, \mu(B_t) \\ &\leqslant \frac{1}{t} (\mathcal{H}^k \llcorner E)(\mathbb{R}^n \setminus E) = 0 \quad \forall\, t > 0\,. \end{aligned}$$

Thus (3.56) follows.

(ii) Let us first prove the left inequality in (3.57), which amounts to prove that

$$\mathcal{H}^k(B) = 0 \text{ if } B := \left\{x \in E : \Theta_k^*(E,x) < 2^{-k}\right\}\,. \tag{3.58}$$

Let $t_j := 2^{-k}(1 - 1/j)$, if $j \geqslant 2$, and

$$B_j := \{x \in E : \Theta_k^*(E,x) \leqslant t_j\}\,.$$

Then, by (3.55) with $\mu = \mathcal{H}^k \llcorner E$ and $B = B_j$,

$$\begin{aligned} \mathcal{H}^k(B_j) &= (\mathcal{H}^k \llcorner B_j)(B_j) \leqslant 2^k t_j\, \mu(B_j) \\ &= 2^k t_j\, \mathcal{H}^k(B_j) \quad \forall\, j \geqslant 2\,. \end{aligned}$$

Since $2^k t_j < 1$ for each $j \geqslant 2$, $\mathcal{H}^k(B_j) = 0$ for each j, which implies

$$\mathcal{H}^k(B) = \mathcal{H}^k(\cup_{j=2}^{\infty} B_j) \leqslant \sum_{j=2}^{\infty} \mathcal{H}^k(B_j) = 0$$

and then (3.58) follows. Finally the proof of the right inequality in (3.57) is similar. Indeed this amounts to prove that

$$\mathcal{H}^k(B) = 0 \text{ if } B := \{x \in E : \Theta_k^*(E,x) > 1\} \ . \tag{3.59}$$

Let $t_j := 1 + 1/j$ and

$$B_j := \{x \in E : \Theta_k^*(E,x) \geqslant t_j\} \ .$$

Then, applying (3.54) , we get that $\mathcal{H}^k(B_j) = 0$ for each j, which yields the desired conclusion.

□

3.4. **Area and coarea formulas in $\mathbb{R}^n$ and some applications.**

Some recalls of linear algebra

Before stating the two main results (Theorems 3.48 and 3.11) of this subsection, we recall some notations and results of linear algebra.

Notation: If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linaar map, we identify L with its transformation matrix M_L, that is the $(m \times n)$- matrix which represents L with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. Moreover, if $n = m$ we define $\det L := \det M_L$.

Definition 3.40. (i) Given $L : \mathbb{R}^n \to \mathbb{R}^m$ linear map, its norm $\|L\|$ is defined by

$$\|L\| := \sup\{|L(v)| : |v| \leqslant 1\} \tag{3.60}$$

(ii) A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is *orthogonal* if

$$(Ox, Oy)_{\mathbb{R}^m} = (x,y)_{\mathbb{R}^n} \quad \forall\, x,\, y \in \mathbb{R}^n.$$

(iii) A linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric* if

$$(Sx, y)_{\mathbb{R}^n} = (x, Sy)_{\mathbb{R}^n} \quad \forall\, x,\, y \in \mathbb{R}^n.$$

(iv) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The *adjoint of* L is the linear map $L^T : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$(x, L^T y)_{\mathbb{R}^n} = (Lx, y)_{\mathbb{R}^m} \quad \forall\, x \in \mathbb{R}^n,\, y \in \mathbb{R}^m.$$

(iv) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then the *rank of* L, denoted rank(L) is the rank of its associated transformation matrix M_L, that is, the maximun number of columns (or rows) of M_L linearly independent.

Remark 3.41. (i) If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, the, L is Lipschitz function and $\|L\| = Lip(L)$.

(ii) If $O : \mathbb{R}^n \to \mathbb{R}^m$ is an orthogonal map, then it is injective and so $n \leqslant m$.

Theorem 3.42 (Polar Decomposition)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then*

(i) *If* $n \leqslant m$ *there are a symmetric linear function* $S : \mathbb{R}^n \to \mathbb{R}^n$ *and an orthogonal linear function* $O : \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$L = O \circ S.$$

Moreover it holds

$$|\mathrm{det} S| = \sqrt{\det(L^T \circ L)}\,.$$

(ii) *If* $n \geqslant m$ *there are a symmetric linear function* $S : \mathbb{R}^m \to \mathbb{R}^m$ *and an orthogonal linear function* $O : \mathbb{R}^m \to \mathbb{R}^n$ *such that*

$$L = S \circ O^T\,.$$

Moreover it holds

$$|\mathrm{det} S| = \sqrt{\det(L \circ L^T)}\,.$$

Proof. See [EG, Sect. 3.2]. □

Definition 3.43 (Jacobian of a linear map)**.** Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function.

(i) If $n \leqslant m$ and assume that $L = O \circ S$ as above. We define the *Jacobian of* L to be

$$\mathbf{J}L := |\mathrm{det} S| = \sqrt{\det(L^T \circ L)}\,.$$

(ii) If $n \geqslant m$ and assume that $L = S \circ O^T$ as above. We define the *Jacobian of* L to be

$$\mathbf{J}L := |\mathrm{det} S| = \sqrt{\det(L \circ L^T)}\,.$$

Remark 3.44. (i) It follows from Theorem 3.42 that the definition of $\mathbf{J}L$ is independent of the particular choice of S and O.

(ii) Clearly $\mathbf{J}L = \mathbf{J}L^T$.

Theorem 3.45 (Binet-Cauchy formula)**.** *Assume that* $n \leqslant m$ *and let* $L : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear map. Then*

$$\mathbf{J}L = \sqrt{\sum_{N \subset M_L} (\mathrm{det} N)^2} \tag{3.61}$$

where the sum is understood over each $(n \times n)$*-submatrix* N *of* $(m \times n)$*- transformation matrix* M_L *of* L*.*

Proof. See [EG, Sect. 3.2]. □

Remark 3.46. From the definition of jacobian and Binet-Cauchy formula (3.61), we can infer that, if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then

$$\mathbf{J}L = 0 \text{ if and only if } \mathrm{rank}(L) < \min\{n, m\}\,.$$

Area formula

We are going to prove that the integral n-dimensional Hausdorff measure $\mathcal{H}^n$ with $n = 1, 2, \ldots, m$ turns out to be the classical *surface* n*-measure* (or, also, n*-volume*) for regular n-dimensional submanifold of $\mathbb{R}^m$. Let us recall that, by Theorems 3.25 and 3.31, we partially met this goal for cases $k = 1$, m. We are going to accomplish the task in the remaining cases by means of the *area formula*.

A possible way for introducing a regular n-dimensional submanifold of $\mathbb{R}^m$ ($n \leqslant m$) is through a (regular) parametrization (see, for instance, [Fe, 3.1.19]). For instance, a n- regular submanifold $\Gamma \subset \mathbb{R}^m$ can be given as follows . Assume that

$$\Gamma = f(A)$$

where $A \subset \mathbb{R}^n$ is a regular bounded subset and $f : \bar{A} \subset \mathbb{R}^n \to \mathbb{R}^m$, called *parametrization of* Γ, satisfies

- $f : A \to \mathbb{R}^m$ is injective;
- $f \in \mathbf{C}^1(\bar{A}; \mathbb{R}^m)$ and $df : A \to \mathbb{R}^m$ has maximal rank.

If this is the case, it well-known that the n- volume of Γ, is defined as

$$\mathrm{n-volume}(\Gamma) := \int_A \mathbf{J}f(x)\, d\mathcal{L}^n(x)\,,$$

where $\mathbf{J}f(x)$ denotes the *Jacobian of f at x*. Jacobian $\mathbf{J}f$ is the corrective factor relating the elements of volumes of the domain and image of f.
Example: (Jacobian for a surface in $\mathbb{R}^3$) Suppose that $n = 2$ and $m = 3$, $\Gamma = f(A)$, with $(s,t) \in \bar{A} \subset \mathbb{R}^2$ and $f = f(s,t) = (f_1(s,t), f_2,(s,t), f_3(s,t)) : \bar{A} \subset \mathbb{R}^2 \to \mathbb{R}^3$ satisfying the previous assumptions. Let $\partial_s f := (\partial_s f_1, \partial_s f_2, \partial_s f_3)$, $\partial_t f := (\partial_t f_1, \partial_t f_2, \partial_t f_3)$ and let N_i ($i = 1,2,3$) denote (all) the 2×2-submatrices of the 3×2 Jacobian matrix of f

$$Df(s,t) := \begin{bmatrix} \partial_s f_1(s,t) & \partial_t f_1(s,t) \\ \partial_s f_2(s,t) & \partial_t f_2(s,t) \\ \partial_s f_3(s,t) & \partial_t f_3(s,t) \end{bmatrix}$$

Then it is well-known by differential geometry that the Jacobian of f at (u,v) equals

$$\mathbf{J}f(s,t) = |\partial_s f \wedge \partial_t f|(s,t) = \sqrt{(\det N_1)^2 + (\det N_2)^2 + (\det N_3)^2}(s,t) \quad (s,t) \in \bar{A}\,,$$

where $v \wedge w$ denotes the exterior product of vectors v, $w \in \mathbb{R}^3$.

Problem: $\mathcal{H}^n(\Gamma) = \mathrm{n-volume}(\Gamma)$?
A positive answer is a particular case of the area formula.

Definition 3.47. Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable at $x \in \mathbb{R}^n$.

(i) If $n \leqslant m$, we define the *Jacobian of f at x* to be

$$\mathbf{J}f(x) := \mathbf{J}df(x) = \sqrt{\det\left(Df(x)^T \cdot Df(x)\right)}\,;$$

(ii) If $n \geqslant m$, we define the *Jacobian of f at x* to be

$$\mathbf{J}f(x) := \mathbf{J}df(x) = \sqrt{\det\left(Df(x) \cdot Df(x)^T\right)}\,;$$

where $Df(x)$ denote the $m \times n$-Jacobian matrix of f at x.

According to the previous definition, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz with $n \leqslant m$, we define as **Jacobian of** f the Borel measurable function $\mathbf{J}f : \mathbb{R}^n \to [0,\infty]$ defined as (3.62)

$$\mathbf{J}f(x) := \begin{cases} \sqrt{\det\left(Df(x)^T \cdot Df(x)\right)} & \text{if } f \text{ is differentiable at } x \\ \infty & \text{if } f \text{ is not differentiable at } x \end{cases} \qquad \text{if } n \leqslant m\,,$$

and
(3.63)

$$\mathbf{J}f(x) := \begin{cases} \sqrt{\det\left(Df(x)\cdot Df(x)^T\right)} & \text{if } f \text{ is differentiable at } x \\ \infty & \text{if } f \text{ is not differentiable at } x \end{cases} \quad \text{if } n \geqslant m\,.$$

Notice that, by Rademacher's theorem, set $\{\mathbf{J}f < \infty\}$ coincides with the set of points $x \in \mathbb{R}^n$ at which f is differentiable and it has full Lebesgue measure in $\mathbb{R}^n$.

Theorem 3.48 (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz with $n \leqslant m$. Then for each $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$*

$$\int_A \mathbf{J}f\, d\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^0\left(A \cap f^{-1}(y)\right) d\mathcal{H}^n(y)\,. \tag{AF}$$

Function $\mathbb{R}^m \ni y \mapsto \mathcal{H}^0\left(A \cap f^{-1}(y)\right) \in \mathbb{N} \cup \{\infty\}$ is called *multiplicity function of f*.

Remark 3.49. (i) Notice that, if $y \notin f(A)$, then $\mathcal{H}^0\left(A \cap f^{-1}(y)\right) = 0$. Hence (AF) can be equivalently written

$$\int_A \mathbf{J}f\, d\mathcal{L}^n = \int_{f(A)} \mathcal{H}^0\left(A \cap f^{-1}(y)\right) d\mathcal{H}^n(y)\,. \tag{3.64}$$

(ii) It follows that, if $n \leqslant m$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and A is bounded, then, by (AF),

$$\mathcal{H}^0\left(A \cap f^{-1}(y)\right) < \infty \quad \mathcal{H}^n\text{-a.e. } y \in \mathbb{R}^m\,.$$

Therefore $f^{-1}(y)$ is at most countable for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$.

Area formula (AF) immediately yields a positive answer to the previous question.

Theorem 3.50 (Area formula for injective maps)**.** *Let $n \leqslant m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an injective Lipschitz function and $A \subset \mathbb{R}^n$ be a measurable set. Then*

$$\mathcal{H}^n(f(A)) = \int_A \mathbf{J}f\, d\mathcal{L}^n \tag{IAF}$$

and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$.

We are now going to give an idea of the proof of the area formula for injective maps, that is Theorem 3.50. Proof of the general area formula, that is Theorem 3.48, can be obtained by using the are formula formula for injective maps (see [Mag, Theorem 8.9]).

We preliminarly need the following fundamental results.

Lemma 3.51 (Measurability of Lipschitz functions images)**.** *If $n \leqslant m$, E is a Lebesgue measurable set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, then $f(E)$ is $\mathcal{H}^n$-measurable in $\mathbb{R}^m$.*

Proof. See [EG, Lemma 2, Sect. 3.3.1] or [Mag, Lemma 8.4]. □

Lemma 3.52 (Area formula for linear functions)**.** *Let $n \leqslant m$ and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then for all $A \subset \mathbb{R}^n$*

$$\mathcal{H}^n(L(A)) = \mathbf{J}L\, \mathcal{L}^n(A) \tag{3.65}$$

Remark 3.53. Notice that that an alternative definition of Jacobian $\mathbf{J}L$ for a linear map $L:\mathbb{R}^n\to\mathbb{R}^m$ $(n\leqslant m)$ can be as value $\mathcal{H}^n(L(Q))$ where $Q=[0,1]^n$ is the unit cube of $\mathbb{R}^n$.

Proof. See [EG, Lemma 1, Sect. 3.3.1] or [Mag, Theorem 8.5]. ☐

Lemma 3.54 (Role of the singular set of a Lipschitz map)**.** *Let $n\leqslant m$ and $f:\mathbb{R}^n\to\mathbb{R}^m$ be Lipschitz. Then*

$$\mathcal{H}^n\left(f(\{x\in\mathbb{R}^n:\,\mathbf{J}f(x)=0\})\right)=0\,.$$

Proof. See [Mag, Theorem 8.7]. ☐

Lemma 3.55 (Lipschitz linearization)**.** *Let $n\leqslant m$ and $f:\mathbb{R}^n\to\mathbb{R}^m$ be Lipschitz. Let us fix (an arbitrary) $t>1$. Denote*

$$F:=\{x\in\mathbb{R}^n:0<\,\mathbf{J}f(x)<\,\infty\}\ ,$$

then there exists a countable disjoint family of Borel sets $(F_h)_h\in\mathbb{N}$ such that

(i) $F=\cup_{h=1}^{\infty}F_h$;
(ii) $f|_{F_h}$ *is injective;*
(iii) *for each h, there exists a symmetric automorphism $S_h:\mathbb{R}^n\to\mathbb{R}^n$ such that $f|_{F_h}\circ S_h^{-1}:\,S_h(F_h)\subset\mathbb{R}^n\to f(F_h)\subset\mathbb{R}^m$ is a bi-Lipschitz map and the following estimates hold: for every x, $y\in F_h$ and $v\in\mathbb{R}^n$*

$$Lip\left(f|_{F_h}\circ S_h^{-1}\right)\leqslant t,\quad Lip\left(S_h\circ(f|_{F_h})^{-1}\right)\leqslant t \tag{3.66}$$

$$\frac{1}{t}\,|S_hv|\leqslant\,|Df(x)v|\leqslant\,t\,|S_hv|, \tag{3.67}$$

$$\frac{1}{t^n}\mathbf{J}S_h\leqslant\,\mathbf{J}f(x)\leqslant\,t^n\,\mathbf{J}S_h. \tag{3.68}$$

Proof. The primitive proof is in [Fe, 3.2.2]; see also [EG, Lemma 3, Sect. 3.3.1] and [Mag, Theorem 8.8]. ☐

Proof of Theorem 3.50.

Step 1: Let us prove that

$$\mathcal{H}^n(f(A))=\,\mathcal{H}^n(f(A\cap F))\,, \tag{3.69}$$

where F is the set in Lemma 3.55. Let us recall that, by Theorem 3.22 and the agreement $\mathcal{L}^n=\,\mathcal{H}^n$ on $\mathbb{R}^n$ (see Theorem 3.31)

$$\mathcal{H}^n(f(E))\leqslant\,Lip(f)^n\,\mathcal{L}^n(E)\quad\forall\,E\subset\mathbb{R}^n\,. \tag{3.70}$$

Thus both sides of (IAF) are zero whenever $\mathcal{L}^n(A)=\,0$. By Rademacher's theorem and Lemma 3.54, we get

$$\begin{aligned}\mathcal{H}^n(f(A\cap F))&\leqslant\,\mathcal{H}^n(f(A))=\,\mathcal{H}^n(f((A\cap F)\cup(f(A\setminus F)))\\&\leqslant\,\mathcal{H}^n(f(A\cap F)+\mathcal{H}^n(f(A\setminus F))\\&\leqslant\,\mathcal{H}^n(f(A\cap F)+\mathcal{H}^n(f(\{\mathbf{J}f=0\}))+\mathcal{H}^n(f(\{\mathbf{J}f=\infty\}))\\&=\,\mathcal{H}^n(f(A\cap F))\,,\end{aligned}$$

which shows (3.69).

Step 2: Let us prove (IAF). By the previous step, we can assume that $A \subset F$. We now fix $t > 1$ and consider the partition $(F_h)_{h\in\mathbb{N}}$ of F given by Lemma 3.55. We see A as the union of the disjoint sets $(F_h \cap A)_h$, so that, by the global injectivity of f, we have that $f(A)$ is the disjoint union of the sets $(f(F_h \cap A))_h$, which are $\mathcal{H}^n$-measurable by Lemma 3.51. Therefore, by Theorem 3.22, the linear case of the area formula (3.65) and (3.66), we find that

$$
\begin{aligned}
\mathcal{H}^n(f(A)) &= \sum_{h=1}^{\infty} \mathcal{H}^n\left(f(A\cap F_h)\right) = \sum_{h=1}^{\infty} \mathcal{H}^n\left(\left(f|_{F_h}\circ S_h^{-1}\right)\left(S_h(A\cap F_h)\right)\right) \\
&\leqslant \sum_{h=1}^{\infty} Lip(f|_{F_h}\circ S_h^{-1})^n\,\mathcal{H}^n\left(S_h(A\cap F_h)\right) \\
&\leqslant t^n \sum_{h=1}^{\infty} \mathbf{J}S_h\,\mathcal{L}^n(A\cap F_h) \\
&\leqslant t^{2n} \sum_{h=1}^{\infty} \int_{A\cap F_h} \mathbf{J}f(x)\,dx = t^{2n}\int_A \mathbf{J}f(x)\,dx \quad \forall\, t > 1\,.
\end{aligned} \tag{3.71}
$$

In a similar way, by analogous argument

$$
\begin{aligned}
\int_A \mathbf{J}f(x)\,dx &= \sum_{h=1}^{\infty}\int_{A\cap F_h}\mathbf{J}f(x)\,dx \leqslant t^n \sum_{h=1}^{\infty}\mathbf{J}S_h\,\mathcal{L}^n(A\cap F_h) \\
&= t^n \sum_{h=1}^{\infty}\mathcal{H}^n\left((S_h\circ(f|_{F_h})^{-1})f(A\cap F_h)\right) \\
&\leqslant t^{2n}\sum_{h=1}^{\infty}\mathcal{H}^n\left(f(A\cap F_h)\right) = t^{2n}\,\mathcal{H}^n\left(f(A)\right) \quad \forall\, t > 1\,.
\end{aligned} \tag{3.72}
$$

We thus prove (IAF) by letting $t \to 1^+$ in (3.71) and (3.72).

Step 3: Let us prove that $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure. By Lemma 3.51, $f(\mathbb{R}^n)$ is $\mathcal{H}^n$-measurable, while (IAF) implies $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ to be locally finite. By Theorem 1.94, $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure.

□

Some applications of the area formula:

(1) (Length of a curve). Assume that $n = 1$, $m \geqslant 1$ and $f : \mathbb{R} \to \mathbb{R}^m$ is an injective Lipschitz function. In this case, the Jacobian matrix of f

$$
Df(t) = f'(t)^T = \begin{bmatrix} f_1'(t) \\ \cdots \\ \cdots \\ f_m'(t) \end{bmatrix}_{m\times 1} \qquad \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}\,,
$$

and according to definition of the Jacobian of f

$$
\mathbf{J}f(t) = \sqrt{\det\left(Df(t)^T\cdot Df(t)\right)} = |f'(t)| \quad \mathcal{L}^1\!-\!a.e.t \in \mathbb{R}\,.
$$

Thus, by (IAF), for a, $b \in \mathbb{R}$ with $a < b$,

$$\mathcal{H}^1(\Gamma) = \int_a^b |f'(t)|\, dt \quad \text{if } \Gamma := f([a,b])\,.$$

(2) (Area of a graph). Assume that $n \geqslant 1$, $m = n+1$ and $u : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. Let $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined as

$$f(x) := (x, u(x)) \quad x \in \mathbb{R}^n\,.$$

Then

$$Df(x) = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ \partial_1 u(x) & \dots & \dots & \dots & \partial_n u(x) \end{bmatrix}_{(n+1)\times n} \qquad \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n\,,$$

and, by applying Binet-Cauchy formula (3.61),

$$\mathbf{J}f(x) = \sqrt{\det\left(Df(x)^T \cdot Df(x)\right)} = \sqrt{1 + |\nabla u(x)|^2} \quad \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n\,. \tag{3.73}$$

Thus, if $A \subset \mathbb{R}^n$ is an open set and $\Gamma = f(A) = \text{graph}(u; A) := \{(x, u(x)) : x \in A\}$,

$$\mathcal{H}^n(\Gamma) = \int_A \sqrt{1 + |\nabla u(x)|^2}\, d\mathcal{L}^n(x)\,. \tag{3.74}$$

Finally let us state a general formula for the change of variables.

Theorem 3.56 (Change of variables)**.** *Let $n \leqslant m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. The for each $\mathcal{L}^n$-integrable function $g : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x)\right] d\mathcal{H}^n(y)\,. \tag{3.75}$$

In particular, if f is injective,

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, d\mathcal{L}^n(x) = \int_{f(\mathbb{R}^n)} g(f^{-1}(y))\, d\mathcal{H}^n(y)\,. \tag{3.76}$$

Proof. See [EG, Theorem 2, Sect. 3.3.3]. □

Remark 3.57. Recall that $f^{-1}(y)$ is at most countable $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$ (see Remark 3.49 (ii)).

Coarea formula

We are now going to presente a far-reaching generalization of Fubini's theorem (see Theorem 1.101), which is very useful in GMT and, more generally, in analysis. Let us begin with an example.

Example: Let $n = 2$ and $Q = [0,1]^2$ and let $L : \mathbb{R}^2 \to \mathbb{R}$ be the linear map $L(x_1, x_2) := x_1$, that is the (orthogonal) projection on the x_1-axis. If $t \in \mathbb{R}$, then

$$L^{-1}(t) = \{(t, x_2) : x_2 \in \mathbb{R}\}\,,$$

that is, $L^{-1}(t)$ is a straight line parallel to the x_2-axis. It is trivial to see that, if we denote $A_t := \{s \in \mathbb{R} : (t,s) \in A\}$, then

$$\mathcal{L}^1(A_t) = \mathcal{H}^1(Q \cap L^{-1}(t)) \quad \forall\, t \in \mathbb{R}\,,$$

and, by Fubini's theorem, we can write that, if $n = 2$, $m = 1$,

$$\mathcal{H}^2(Q) = \mathcal{L}^2(Q) = \int_{\mathbb{R}} \mathcal{L}^1(A_t)\, dt = \int_{\mathbb{R}} \mathcal{H}^{n-m}(Q \cap L^{-1}(t))\, dt\,. \tag{3.77}$$

Let us now consider the linear map $L(x_1, x_2) = -x_1 + x_2$. If $t \in \mathbb{R}$, then

$$L^{-1}(t) = \{(x_1, x_1 + t) : x_1 \in \mathbb{R}\}\ ,$$

that is $L^{-1}(t)$ is still a straight line, but it is not parallel to the coordinate axes. In particular identity (3.77) no more holds. Indeed, we can get, by a simple calculation, that

$$\mathcal{H}^1(Q \cap L^{-1}(t)) = \begin{cases} \sqrt{2}(1 - |t|) & \text{if } |t| \leqslant 1 \\ 0 & \text{if } |t| > 1 \end{cases}$$

and

$$\int_{\mathbb{R}} \mathcal{H}^1(Q \cap L^{-1}(t))\, dt = \sqrt{2} = \sqrt{2}\, \mathcal{H}^2(Q)\,.$$

Problem: Let $n \geqslant m$ and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, find out a non negative factor $c(L)$ such that

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(y))\, d\mathcal{L}^m(y) = c(L)\, \mathcal{L}^n(A) \quad \forall\, A \subset \mathbb{R}^n\,. \tag{3.78}$$

Theorem 3.58 (Coarea for linear maps)**.** *Let $n \geqslant m$ and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then* (3.78) *holds with*

$$c(L) := \mathbf{J}L = \sqrt{\det(L \circ L^T)}$$

where $L^T : \mathbb{R}^m \to \mathbb{R}^n$ denotes the adjoint linear map of L.

Proof. See [EG, Lemma 1, Sect. 3.4.1]. □

Theorem 3.59 (Corea formula)**.** *Let $n \geqslant m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. The for each $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$*

$$\int_A \mathbf{J}f(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y)\,,$$

where $\mathbf{J}f$ is the Jacobian factor defined in (3.63)*.*

Proof. See [EG, Theorem 1, Sect. 3.4.1]. □

Remark 3.60. Applying the coarea formula to set $A := \{x \in \mathbb{R}^n : \mathbf{J}f(x) = 0\}$, we get that

$$\mathcal{H}^{n-m}\left(\{\mathbf{J}f = 0\} \cap f^{-1}(y)\right) = 0 \quad \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m\,. \tag{WMS}$$

This is a weak variant of Morse-Sard's theorem which asserts

$$\{\mathbf{J}f = 0\} \cap f^{-1}(y) = \emptyset \quad \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m\,, \tag{MS}$$

provided that $f \in \mathbf{C}^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k = 1 + n - m$. Observe, however, that (WMS) only requires that f be Lipschitz.

Theorem 3.61 (Change of variables formula)**.** *Under the same assumptions of the Coarea Formula. Then for each $\mathcal{L}^n$-measurable function $g : \mathbb{R}^n \to \mathbb{R}$*

(i) $g|_{f^{-1}(y)}$ *is $\mathcal{H}^{n-m}$-summable $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$.*
(ii)

$$\int_{\mathbb{R}}^n g(x)\,\mathbf{J}f(x)\,d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\int_{f^{-1}(y)} g(x)\,d\mathcal{H}^{n-m}(x)\right] d\mathcal{L}^m(y)\,,$$

Proof. See [EG, Theorem 2, Sect. 3.4.3]. □

3.5. **Extensions to metric spaces.**

About Hausdorff measures and length measure

Let (X, d) be a metric space. A *curve* of X is a continuous function $\gamma : [a, b] \to X$; its *support* is the subset $\Gamma = \gamma([a, b]) \subset X$ and, in this case, γ is called a *parametrization* of Γ. Given a curve $\gamma : [a, b] \to X$ and a subinterval $[c, d] \subseteq [a, b]$, we define *variation of γ on* $[c, d]$

$$\mathrm{Var}(\gamma; [c, d)) := \sup\left\{\sum_{i=1}^m d(\gamma(t_i), \gamma(t_{i-1})) : t_0 = c < t_1 < \cdots < t_m = d\right\} \in [0, \infty]\,.$$

By analogy with the case where $X = \mathbb{R}^n$ and d is the Euclidean distance, the quantity $\mathrm{Var}(\gamma; [c, d))$ represents the length (with respect to the metric d) of curve γ over $[c, d]$.

Therehore, given a subinterval $[c, d] \subseteq [a, b]$, we define the *length of γ over* $[c, d]$ as

$$l(\gamma; [c, d]) := \mathrm{Var}(\gamma; [c, d))\,. \tag{3.79}$$

Thus we can define as *length of γ* the quantity

$$l(\gamma) := l(\gamma; [a, b])\,.$$

We say that γ is *rectifiable* if $l(\gamma) < \infty$. It is easy to see that, if $\gamma : [a, b] \to X$ is a curve, then

$$l(\gamma; [c, d]) \geqslant d(\gamma(c), \gamma(d)), \text{whenever } a \leqslant c \leqslant d \leqslant a\,; \tag{3.80}$$

and

$$l(\gamma; [a, b]) = l(\gamma; [a, c]) + l(\gamma; [c, b]),\ \text{whenever } a \leqslant c \leqslant b\,. \tag{3.81}$$

Theorem 3.62 (Classical length and $\mathcal{H}^1$)**.** *Let $\gamma : [a, b] \to X$ be a curve and denote $\Gamma = \gamma([a, b])$ its support. Then*

$$\mathcal{H}^1(\Gamma) \leqslant \mathcal{S}^1(\Gamma) \leqslant l(\gamma)$$

and equality holds if γ is injective, where $\mathcal{S}^1$ denotes the 1-dimensional spherical Hausdorff measure defined in Remark 3.5.

Proof. Proof's strategy is similar to the one of Theorem 3.25 by means of suitable changes (see [AT, Theorem 4.4.2] and [SC, Theorem 2.29]). □

About Hausdorff dimension in a metric measure space

An useful criterion to estimate the Hausdorff dimension of a metric measure space is the following.

Theorem 3.63. *Let* (X, d, μ) *be a metric measure space with* (X, d) *separable. Suppose that there exist constants* $c > 1$ *and* $Q > 0$ *such that*

$$\frac{1}{c} r^Q \leqslant \mu(B(x,r)) \leqslant c\, r^Q \quad \forall\, 0 < r < d(X)\,. \tag{3.82}$$

Then there exists a constant $c' > 1$ *such that*

$$\frac{1}{c'} \mathcal{H}^Q(E) \leqslant \mu(E) \leqslant c'\, \mathcal{H}^Q(E)\,, \tag{3.83}$$

for each Borel set $E \subset X$*. In particular*

$$\mathrm{Hdim}(X) = Q\,.$$

Proof. See, for instance, [SC, Theorem 2.26]. □

Metric measure spaces where formula (3.82) holds are called *Ahlfors regular of dimension Q*. By (3.83), we can replace μ by the Hausdorff Q-measure in an Q-regular space without essential loss of information.

About Lipschitz maps

Extension's problem for Lipschitz maps acting between metric spaces is an issue of the current research in analysis in metric spaces (see [AT, Chap. 3], [He, Chap. 6] and [He2]). The proof of Kirszbraun's theorem depends crucially on very euclidean properties of the domain space $\mathbb{R}^n$. Hence its proof cannot be extended to general metric spaces.

Differentiability for Lipschitz functions $f : \mathbb{R}^k \to (X, d)$ has been studied. In particular Rademacher-type theorems have been obtained (see [AK]).

A very important Rademacher type-result for real valued Lipschitz functions defined on a metric measure space (X, d, μ), that is Lipschitz functions $f : (X, d, \mu) \to \mathbb{R}$, has been obtained by Cheeger [Ch] and it has been a seminal paper for the developement of analysis in metric measure spaces (see, for instance, [AG] and the references therein).

About Lipschitz maps and Hausdorff measures

Notice that the conclusion of Theorem 3.22 holds for each Lipschitz map $f : (X, d) \to (Y, \varrho)$.

Notice also that for every isometric embedding $f : (X, d) \to (Y, \varrho)$, that is a mapping satisfying $\varrho(f(x), f(y)) = d(x, y)$, by definition of Hausdorff measure, it follows that

$$\mathcal{H}^s(f(A)) = \mathcal{H}^s(A) \quad \forall\, A \subset X\,.$$

About area and coarea formulas

Area and coarea formulas have been obtained respectively for Lipschitz functions $f : \mathbb{R}^k \to (X, d)$ and $f : (X, d) \to \mathbb{R}^k$ (see [AK]).

4. Rectifiable sets and blow-ups of Radon measures ([AFP, Mag, Ma])

Motivation: the introduction of the notion of rectifiable set in $\mathbb{R}^n$, which is a set smooth in a certain measure-theoretic sense. It is an extension of the idea of a rectifiable curve to higher dimensions and it has many of the desirable properties of smooth manifolds, including tangent spaces that are defined almost everywhere. Rectifiable sets are fundamental objects of study in geometric measure theory.

k-dimensional planes and the orthogonal group in $\mathbb{R}^n$.

Let us begin to recall the notion of k-dimensional plane in $\mathbb{R}^n$ and some their properties.

We simply mean by a *k-dimensional plane π in $\mathbb{R}^n$* a k-dimensional subspace of $\mathbb{R}^n$. We will denote by

$$G(n, k)$$

the class of k-dimensional planes of $\mathbb{R}^n$, which is also called *Grassmannian manifold of k dimensional subspaces of $\mathbb{R}^n$*.

If π is a k-dimensional plane we denote by $\pi^\perp$ the $(n-k)$-plane orthogonal to π, that is

$$\pi^\perp := \{x \in \mathbb{R}^n : (x, y)_{\mathbb{R}^n} = 0 \; \forall \, y \in \pi\} .$$

Notice also that, if $\pi \subset \mathbb{R}^n$ is a k-plane and $\{v_1, \dots, v_k\}$ is an orthonormal basis of π then the map $I_\pi : \mathbb{R}^k \to \mathbb{R}^n$

$$I_\pi(x) := \sum_{j=1}^{k} x_j \, v_j \quad \text{if } x \in \mathbb{R}^k \tag{4.1}$$

defines an (injective) orthogonal map such that $I_\pi(\mathbb{R}^k) = \pi$.

Given a k-dimensional plane π in $\mathbb{R}^n$, we denote by $P_\pi : \mathbb{R}^n \to \mathbb{R}^n$ and $P_\pi^\perp : \mathbb{R}^n \to \mathbb{R}^n$ respectively the orthogonal projections of $\mathbb{R}^n$ onto π and $\pi^\perp$. In particular it turns out that $P_\pi^\perp = P_{\pi^\perp}$.

Tipically we identify π with P_π and we endow $G(n, k)$ by the distance

$$|\pi_1 - \pi_2|_G := \|P_{\pi_1} - P_{\pi_2}\| = \sup_{v \in \mathbf{S}^{n-1}} |P_{\pi_1}(v) - P_{\pi_2}(v)| \text{ if } \pi_i \in G(n, k) \; i = 1, 2 \, , \tag{4.2}$$

that is $\| \cdot \|$ denotes the norm in (3.60).

Exercise: Prove that

$$(G(n, k), | \cdot |_G) \text{ is a compact metric space.} \tag{4.3}$$

The *orthogonal group $O(n)$ of $\mathbb{R}^n$* consists of all linear orthogonal maps $L : \mathbb{R}^n \to \mathbb{R}^n$ according to Definition 3.40 (ii), that is linear maps preserving the inner product,

$$(L(x), L(y))_{\mathbb{R}^n} = (x, y)_{\mathbb{R}^n} \quad \forall \, x, y \in \mathbb{R}^n \, ,$$

or equivalently L is an isometry,

$$|L(x) - L(y)| = |x - y| \quad \forall\, x, y \in \mathbb{R}^n\,.$$

It is easy to see that $O(n)$ turns out to be a group with composition as group law. The members of $O(n)$ consist of rotations and rotations composed with a reflexion over some hyperplane. Another way to view them is to observe that they map orthonormal basis to orthonormal basis, and conversely given two orthonormal bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ of $\mathbb{R}^n$ one can define $L \in O(n)$ by setting $L(u_i) = v_i$ and extending linearly.

One of the basic properties of $O(n)$ is that it *acts transitively on* $\mathbf{S}^{n-1}$: for any $x, y \in \mathbf{S}^{n-1}$ there exists $L \in O(n)$ such that $L(x) = y$.

In the case $n = 2$, $0(2)$ is very simple. It consists of rotations around the origin and of rotations composed with the reflexion over the x-axis.

By simple linear algebra the action of $O(n)$ on $G(n, k)$ is distance-preserving:

$$|L(\pi_1) - L(\pi_2)|_G \;=\; |\pi_1 - \pi_2|_G \quad \forall\, L \in O(n),\ \pi_i \in G(n, m)\, i = 1, 2\,.$$

Also $O(n)$ transitively acts on $G(n, k)$: for $\pi_i \in G(n, k)$, $i = 1, 2$, there is $L \in O(n)$ such that $L(\pi_1) = \pi_2$, that is π_1 and π_2 are *isometric*. To see this, take orthonormal bases for π_1 and π_2, complete them to orthonormal bases of $\mathbb{R}^n$, and choose $L \in O(n)$ which maps one of these onto the other.

In particular it follows, if $B_1 = B(1)$ denotes the unit closed ball in $\mathbb{R}^n$ centered at 0 and π is a k-dimensional plane in $\mathbb{R}^n$, then set $\pi \cap B_1$ is isometric to the unit closed ball in $\mathbb{R}^k$ and so, by Remark 3.24,

$$\mathcal{H}^k(\pi \cap B_1) = \alpha_k^*\,. \tag{4.4}$$

Analogously, set $\pi \cap \partial B_1$ is isometric to the unit sphere in $\mathbb{R}^k$ and so

$$\mathcal{H}^k(\pi \cap \partial B_1) = 0\,, \tag{4.5}$$

Regular surfaces and rectifiable sets.

Let us now recall the classical notion of regular surface in $\mathbb{R}^n$.

Definition 4.1. Given $k \in \mathbb{N}$, $1 \leqslant k \leqslant n - 1$, $m \geqslant 1$, we shall say that $\Gamma \subset \mathbb{R}^n$ is a k-dimensional (embedded) surface (or submanifold) of class $\mathbf{C}^m$ in $\mathbb{R}^n$ (or a $\mathbf{C}^m$-hypersurface when $k = n - 1$) if for every $x \in \Gamma$ there exist an open neighborhood $V \subset \mathbb{R}^n$ of x, an open set $U \subset \mathbb{R}^k$ and a bijection $f : U \to V \cap \Gamma$ with $f \in \mathbf{C}^m(U)$ and $df(x) : \mathbb{R}^k \to \mathbb{R}^n$ is injective for each $x \in U$. Each map f is called a *coordinate mapping of* Γ.

Remark 4.2. (i) It is well-known from standard calculus that $df(x) : \mathbb{R}^k \to \mathbb{R}^n$ is injective if and only if $\mathbf{J}f(x) > 0$, where $\mathbf{J}f(x)$ denotes the Jacobian of f at x (see Definition 3.47 and Remark 3.46).

(ii) Notice that, in this way, Γ is relatively open in $\mathbb{R}^n$, and can be covered by countably many images $f(U)$, with f and U as above.

The notion of rectifiable set is just a generalization of Remark 4.2 (ii) to the measure-theoretic setting.

Definition 4.3. Let $1 \leqslant k \leqslant n - 1$ be integers and let $\Gamma \subset \mathbb{R}^n$.

(i) We say that Γ is *countably $\mathcal{H}^k$-rectifiable* if

(CR) $$\Gamma = \Gamma_0 \cup (\cup_{i=1}^{\infty} f_i(A_i)) \ ,$$

where $\mathcal{H}^k(\Gamma_0) = 0$, $A_i \subset \mathbb{R}^k$ are $\mathcal{L}^k$-measurable and $f_i : A_i \to \mathbb{R}^n$ are Lipschitz functions for $i = 1, 2, \ldots$.

(ii) We say that Γ is *locally $\mathcal{H}^k$-rectifiable* if Γ is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(\Gamma \cap K) < \infty$ for each compact $K \subset \mathbb{R}^n$.

(iii) We say that Γ is *$\mathcal{H}^k$-rectifiable* if Γ is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(\Gamma) < \infty$.

Remark 4.4. Since Lipschitz functions defined on subsets of $\mathbb{R}^k$ can be extended to all of $\mathbb{R}^k$, keeping a control of the Lipschitz constant (see Theorem 3.18), then we can equivalently say that Γ is countably k-rectifiable if it is $\mathcal{H}^k$-measurable and

(CR*) $$\Gamma \subset \Gamma_0 \cup (\cup_{i=1}^{\infty} \Gamma_i) \ ,$$

where $\mathcal{H}^k(\Gamma_0) = 0$, $\Gamma_i := f_i(\mathbb{R}^k)$ and $f_i : \mathbb{R}^k \to \mathbb{R}^n$ are Lipschitz functions for $i = 1, 2, \ldots$.

Remark 4.5. Observe that the rectifiability is a *metric* notion since it depends on the notion of Lipschitz maps acting between metric spaces (see section IV.4).

Historical notes: The concepts of rectifiability were first introduced for one-dimensional sets in $\mathbb{R}^2$ by Besicovitch [Be3] in 1928 and then subsequently studied in papers [Be4, Be5] for $k = 1$ and $n = 2$. Federer extended the study for general k and n in 1947 and those results are contained in his celebrated monograph [Fe].

Example 4.6 (Lipschitz k-graph)**.** Let π be a k-plane and let $\phi : \pi \to \pi^{\perp}$ be a Lipschitz function.We call *graph of ϕ* the set

$$\Gamma = \text{graph}(\phi) := \{z + \phi(z) : z \in \pi\} \subset \mathbb{R}^n \, .$$

Let us define $f : \mathbb{R}^k \to \mathbb{R}^n$

$$f(x) := I_\pi(x) + \phi(I_\pi(x)) \quad x \in \mathbb{R}^k \, ,$$

where $I_\pi : \mathbb{R}^k \to \mathbb{R}^n$ is the map defined in (4.1). It is easy to check that f is still Lipschitz and, from Theorem 3.22, we get that Γ is locally $\mathcal{H}^k$-rectifiable. In particular any compact subset of Γ is $\mathcal{H}^k$-rectifiable.

Remark 4.7. Observe that that countable $\mathcal{H}^k$- rectifiability (CR*) is equivalent to the seemingly stronger requirement that sets Γ_i $(i = 1, 2, \ldots)$ is a Lipschitz k-graphs (see [AFP, Proposition 2.76]). Using Whitney's extension theorem (see Theorem 3.14) to approximate Lipschitz functions by $\mathbf{C}^1$ functions, it could be possible to show that Γ_i $(i = 1, 2, \ldots)$ can be a $\mathbf{C}^1$ k-graphs (see [Fe, 3.1.16]). However, since Lipschitz functions are more flexible than $\mathbf{C}^1$ functions in many typical constructions of geometric measure theory (see Mc Shane's extension theorem 3.16), the use of Lipschitz functions is preferred.

In this chapter we are going to study some geometric properties of locally $\mathcal{H}^k$-rectifiable sets, such as the existence $\mathcal{H}^k$-a.e. of a tangent plane in a suitable sense and a their characterization in terms of measure-theoretic properties. We will strictly follow the arguments in [Mag, Chap. 10].

Observe that, whenever Γ is a countably $\mathcal{H}^k$-rectifiable, then $\mathcal{H}^k \llcorner \Gamma$ is a regular Borel outer measure. However, from Theorem 1.94, $\mathcal{H}^k \llcorner \Gamma$ is a Radon measure if and only if Γ is locally $\mathcal{H}^k$-rectifiable. Therefore, it is under the assumption of local $\mathcal{H}^k$-rectifiability on Γ that we have a natural identification between Γ and a Radon measure μ. In turn, as seen in Example 1.111, this identification lies at the basis of the measure-theoretic formulation of the notion of tangent space. Indeed, if Γ is locally $\mathcal{H}^k$-rectifiable and $\mu = \mathcal{H}^k \llcorner \Gamma$, then for $\mathcal{H}^k$-a.e. $x \in \Gamma$ there exists a k-dimensional plane π_x in $\mathbb{R}^n$ such that the blow-ups $\mu_{x,r}$ of μ at x weak* converge to $\mathcal{H}^k \llcorner \pi_x$ as $r \to 0^+$, that is

$$\mathcal{H}^k \llcorner \left(\frac{\Gamma - x}{r} \right) \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x \text{ as } r \to 0^+ . \tag{4.6}$$

A crucial fact is that the converse also holds true: if μ is a Radon measure on $\mathbb{R}^n$ concentrated on a Borel set Γ and such that for every $x \in \Gamma$ there exists a k-dimensional plane π_x such that the k-dimensional blow- ups of μ have the property that

$$\mu_{x,r} = \frac{(\Phi_{x,r})_{\#}\mu}{r^k} \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x, \text{ as } r \to 0^+, \tag{4.7}$$

then Γ is locally $\mathcal{H}^k$ rectifiable and $\mu = \mathcal{H}^k \llcorner \Gamma$ where $\Phi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ is the map defined by

$$\Phi_{x,r}(y) := \frac{y - x}{r} \quad y \in \mathbb{R}^n .$$

4.1. **Rectifiable sets of $\mathbb{R}^n$ and their decomposition in regular Lipschitz images.** In this section we are going to replace the decomposition of a countably $\mathcal{H}^k$-rectifiable set, provided in (CR), with a "good decompostiton" composed or "regular" Lipschitz parametrization maps $f_i : A_i \subset \mathbb{R}^k \to \mathbb{R}^n$. Let us begin to introduce the notion of regular parametrization Lipschitz map.

Definition 4.8 (Regular Lipschitz image)**.** Given a Lipschitz map $f : \mathbb{R}^k \to \mathbb{R}^n$ and a compact set $E \subset \mathbb{R}^k$, we say that the pair (f, E) defines a *regular Lipschitz image* $f(E)$ in $\mathbb{R}^n$ if

- (i) f is injective and differentiable on E, with $\mathbf{J}f(x) > 0$ for each $x \in E$;
- (ii) $\mathcal{L}^k$-a.e. $x \in E$ is a point of density 1 for E;
- (iii) $\mathcal{L}^k$-a.e. $x \in E$ is a Lebesgue point of Df.

Example 4.9. A k-regular $\mathbf{C}^1$ surface $\Gamma \subset \mathbb{R}^n$ can be seen as countable union of regular Lipschitz images. Indeed, according to Definition 4.1 and Remark 4.2, Γ can be covered by a countable union of relatively compact open sets $(V_i)_i$ of $\mathbb{R}^n$ for which there exists a countable family of coordinate mappings $f_i : U_i \to \Gamma \cap V_i$, which are bijective, with $(U_i)_i$ relatively compact open sets of $\mathbb{R}^k$, $f_i \in \mathbf{C}^1(\bar{U}_i; \mathbb{R}^n)$, $df_i(x) : \mathbb{R}^k \to \mathbb{R}^n$ is injective for each $x \in \bar{U}_i$ and $\mathbf{J}f_i(x) > 0$ for each $x \in \bar{U}_i$. Moreover, since each map $f_i : \bar{U}_i \to \mathbb{R}^n$ is a Lipschitz function, it can be extended, as a Lipschitz function, to the whole $\mathbb{R}^k$ (see Corollary 3.17) . Therefore the sequence of couples (f_i, E_i), with $E_i := \bar{U}_i$ yield a countable union of regular Lipschitz images $f(E_i)$.

Remark 4.10. In particular, we immediately deduce from (ii) and (iii) of Definition 4.8 that

$$\lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{\alpha_k^* \, r^k} = 1, \quad \lim_{r \to 0^+} \frac{1}{r^k} \int_{B(x,r)} |\mathbf{J}f(x) - \mathbf{J}f(y)| \, dy = 0$$

for every $x \in E$. Indeed, if x is a Lebesgue point of the Jacobian matrix of f, Df, then x is a Lebesgue point of $\mathbf{J}f$, since $Df \in L^\infty(\mathbb{R}^k; M_{n,k})$ and the map $\mathcal{L}(\mathbb{R}^k;\mathbb{R}^n) \ni L \mapsto \mathbf{J}L$ is continuous, where $M_{n,k}$ and $\mathcal{L}(\mathbb{R}^k;\mathbb{R}^n)$ denote respectively the class of real matrices with n rows and k columns and the space of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^n$.

We now show that we can always decompose a countably $\mathcal{H}^k$- rectifiable set by means of (almost flat) regular Lipschitz images (see also Lemma 3.11).

Theorem 4.11 (Decomposition of rectifiable sets)**.** *If Γ is countably $\mathcal{H}^k$- rectifiable in $\mathbb{R}^n$ and $t > 1$, then there exist a Borel set $\Gamma_0 \subset \mathbb{R}^n$, countably many Lipschitz maps $f_h : \mathbb{R}^k \to \mathbb{R}^n$ and compact sets $E_h \subset \mathbb{R}^k$ such that*

$$\Gamma = \Gamma_0 \cup (\cup_{h=1}^\infty f_h(E_h)) , \quad \mathcal{H}^k(\Gamma_0) = 0 . \tag{4.8}$$

Each pair (f_h, E_h) defines a regular Lipschitz image, with $Lip(f_h) \leqslant t$ and

$$t^{-1}|x-y| \leqslant |f_h(x) - f_h(y)| \leqslant t\,|x-y|, \tag{4.9}$$

$$t^{-1}\,|v| \leqslant |Df_h(x)v| \leqslant t\,|v|, \tag{4.10}$$

$$t^{-k} \leqslant \mathbf{J}f_h(x) \leqslant t^k, \tag{4.11}$$

for every x, $y \in E_h$ and $v \in \mathbb{R}^k$.

Before the proof of Theorem 4.8 we need the following result.

Exercise: Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a Lipschitz map. Then there exists a Borel set $D \subset \mathbb{R}^k$ such that $\mathcal{L}^k(\mathbb{R}^k \setminus D) = 0$ and, for each $x \in D$, f is differentiable at x.

(**Hint:** Use Radamacher's theorem (see Theorem 3.19 and that $\mathcal{L}^k$ is a Borel regular o.m.)

Proof of Theorem 4.8. Assume that

$$\Gamma = \Gamma_0 \cup (\cup_{i=1}^\infty f_i^*(A_i)) , \tag{4.12}$$

where $\mathcal{H}^k(\Gamma_0) = 0$, $A_i \subset \mathbb{R}^k$ are $\mathcal{L}^k$-measurable and $f_i^* : A_i \to \mathbb{R}^n$ are Lipschitz functions for $i \in \mathbb{N}$. Without loss of generality, by the Borel regularity of $\mathcal{L}^k$, Theorem 3.22 and Kirszbraun's theorem (see Theorem 3.18), we can also assume that, for each i, A_i is a Borel set and $f_i^* : \mathbb{R}^k \to \mathbb{R}^n$ is a Lipschitz function.

Let us divide the proof in three steps.

1st step: We can suppose that, for each i, f_i^* is differentiable at each point $x \in A_i$ and

$$A_i \subset F_i := \left\{x \in \mathbb{R}^k : 0 < \mathbf{J}f_i^*(x) < \infty\right\} . \tag{4.13}$$

Indeed, let denote $\tilde{A}_i$ the subset of points $x \in A_i$ such that f_i^* is differentiable at x. By the previous exercise we can assume that $\tilde{A}_i$ is a Borel set for each i. Then, since, by Rademacher theorem),

$$\mathcal{L}^k(A_i \setminus \tilde{A}_i) = 0 ,$$

it follows that, by Theorem 3.22,

$$\mathcal{H}^k\left(f_i^*\left(A_i \setminus \tilde{A}_i\right)\right) = 0 . \tag{4.14}$$

By the area formula (see (AF)), it turns out that

$$\mathcal{H}^k\left(f_i^*\left(\tilde{A}_i \setminus F_i\right)\right) = 0 \text{ for each } i\,. \tag{4.15}$$

Therefore, by (4.14) and (4.15), we are allowed to use in (4.12) functions $f_i^* : \tilde{A}_i \cap F_i \to \mathbb{R}^n$ and we get the desired conclusion.

2nd step: By the previous step and applying the Lipschitz linearization to each function f_i^* (see Lemma 3.55), we get that, for each $t > 1$ there exists a sequence of disjoint Borel sets $(A_h^{(i)})_h$ such that

$$A_i = \cup_{h=1}^{\infty} A_h^{(i)}\,; \tag{4.16}$$

$$f_i^*|_{A_h^{(i)}} \text{ is injective}\,; \tag{4.17}$$

for each h, there exists a symmetric automorphism $S_h \equiv S_h^{(i)} : \mathbb{R}^k \to \mathbb{R}^k$ such that $f_i^*|_{A_h^{(i)}} \circ S_h^{-1} : S_h(A_h^{(i)}) \subset \mathbb{R}^k \to f_i^*(A_h^{(i)}) \subset \mathbb{R}^n$ is a bi-Lipschitz map and the following estimates hold: for every $x,\, y \in A_h^{(i)}$ and $v \in \mathbb{R}^n$

$$Lip\left(f_i^*|_{A_h^{(i)}} \circ S_h^{-1}\right) \leqslant t, \quad Lip\left(S_h \circ (f_i^*|_{A_h^{(i)}})^{-1}\right) \leqslant t \tag{4.18}$$

$$\frac{1}{t}\,|S_h v| \leqslant |Df_i^*(x)v| \leqslant t\,|S_h v|, \tag{4.19}$$

$$\frac{1}{t^n}\mathbf{J}S_h \leqslant \mathbf{J}f(x) \leqslant t^n\,\mathbf{J}S_h\,. \tag{4.20}$$

Denote, for each i and h,

$$G_h^{(i)} := S_h\left(A_h^{(i)}\right) \subset \mathbb{R}^k, g_h^{(i)} := f_i^*|_{A_h^{(i)}} \circ S_h^{-1} : G_h^{(i)} \to \mathbb{R}^n\,.$$

Then, by Kirszbraun's theorem , we can assume that $g_h^{(i)} : \mathbb{R}^k \to \mathbb{R}^n$ is still Lipschitz with $Lip\left(g_h^{(i)}\right) \leqslant t$ and, by Rademacher's theorem, $g_h^{(i)}$ is differentiable at $\mathcal{L}^k$-a.e. $x \in \mathbb{R}^k$ and

$$Dg_h^{(i)}(x) = Df_i^*(x) \cdot S_h^{-1} \quad \text{for } \mathcal{L}^k\text{-a.e. } x \in G_h^{(i)}\,.$$

Therefore, by (4.16)- (4.20), and arguing again as in the first step, it follows that

$$\text{(4.8)-(4.11) hold with } f_h \equiv g_h^{(i)},\, E_h \equiv G_h^{(i)}\,. \tag{4.21}$$

3rd step: Let us relabel sequences $(g_h^{(i)})_{i,h}$ and $(G_h^{(i)})_{i,h}$ respectively by sequences $(\tilde{f}_h)_h$ and $(\tilde{E}_h)_h$. Therefore, by (4.21), we have that (4.8)-(4.11) hold with $f_h \equiv \tilde{f}_h$ and $E_h \equiv \tilde{E}_h$. We have only to show that we can modify respectively the sequence of functions $(\tilde{f}_h)_h$ and the one of sets $(E_h^*)_h$ by a sequence of functions $(f_h)_h$ and one compact sets $(E_h)_h$ in order that they still satisfies (4.8)-(4.11) and couple (f_h, E_h) induces a regular Lipschitz image for each h. For a given h, denote

$$\tilde{E}_m^{(h)} := \tilde{E}_h \cap B(m) \text{ if } m \in \mathbb{N}\,.$$

where $B(m)$ denotes the closed ball of $\mathbb{R}^k$ centered at 0 with radius m.

By the approximation of Radon measures by compact sets from below (see Theorem 1.14), for each h, m, there exists an increasing sequence of compact sets $(K_{mj}^{(h)})_j$ such that

$$K_{mj}^{(h)} \subset \tilde{E}_m^{(h)} \text{ and } \mathcal{L}^k\left(\tilde{E}_m^{(h)} \setminus K_{mj}^{(h)}\right) < 1/j \; \forall j\,. \tag{4.22}$$

In particular, since

$$\mathcal{L}^k\left(\tilde{E}_m^{(h)} \setminus \cup_{j=1}^{\infty} K_{mj}^{(h)}\right) = 0\,, \text{ for each } h,\, m\,,$$

by the area formula

$$\mathcal{H}^k\left(\tilde{f}_h\left(\tilde{E}_m^{(h)} \setminus \cup_{j=1}^{\infty} K_{mj}^{(h)}\right)\right) = 0 \text{ for each } h,\, m\,. \tag{4.23}$$

Let us denote, for given h, m and j,

$$f_{mj}^{(h)} := \tilde{f}_h \text{ and } E_{mj}^{(h)} := K_{mj}^{(h)}\,;$$

by relabelling sequences $(f_{mj}^{(h)})_{h,m,j}$ and $(E_{mj}^{(h)})_{h,m,j}$, we get sequences $(f_h)_h$ and $(E_h)_h$ respectively which still satisfy (4.8)-(4.11). Let us now prove that, for each h, couple (f_h, E_h) is a regular Lipschitz image. Observe that each E_h is compact and, by the density of a set (see Corollary 2.18) and Lebesgue-Besicovitch differentiation theorem (see Theorem 2.16),we can assume that, for each h and $\mathcal{L}^k$-a.e. $x \in E_h$

$$\lim_{r\to 0^+} \frac{\mathcal{L}^k(E_h \cap B(x,r))}{\alpha_k^*\, r^k} = 1, \qquad x \text{ is a Lebegue point of } Df_h\,.$$

Thus conditions (ii) and (iii) of Definition 4.8 hold. Moreover, by (4.9) and (4.11), condition (i) of Definition 4.8 holds, too.

□

4.2. **Approximate tangent planes to rectifiable sets.** Theorem 4.11 allows us to prove the existence (in a measure-theoretic sense) of tangent spaces to rectifiable sets.

Let us come back to the approximate tangent plane. Define

$$\Phi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n \text{ as } \Phi_{x,r}(y) := \frac{y-x}{r},\; y \in \mathbb{R}^n,$$

so that, if μ is a Radon measure on $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is a Borel set, then

$$\frac{(\Phi_{x,r})_\# \mu(E)}{r^k} = \frac{\mu(x + rE)}{r^k}\,. \tag{4.24}$$

Theorem 4.12 (Existence of approximate tangent spaces)**.** *If $\Gamma \subset \mathbb{R}^n$ is a locally $\mathcal{H}^k$-rectifiable set, then for $\mathcal{H}^k$-a.e. $x \in \Gamma$ there exists a unique k- dimensional plane π_x such that, as $r \to 0^+$,*

$$\frac{(\Phi_{x,r})_\#(\mathcal{H}^k \llcorner \Gamma)}{r^k} = \mathcal{H}^k \llcorner \left(\frac{\Gamma - x}{r}\right) \stackrel{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x\,, \tag{4.25}$$

that is

$$\lim_{r\to 0^+} \frac{1}{r^k} \int_\Gamma \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \int_{\pi_x} \varphi(y)\, d\mathcal{H}^k(y) \quad \forall \varphi \in \mathbf{C}_c^0(\mathbb{R}^n)\,. \tag{4.26}$$

In particular

$$\exists\,\Theta_k(\Gamma,x) := \lim_{r\to 0^+} \frac{\mathcal{H}^k(\Gamma\cap B(x,r))}{\alpha_k^*\, r^k} = 1 \quad \textit{for } \mathcal{H}^k\textit{-a.e. } x\in\Gamma\,. \tag{4.27}$$

Remark 4.13. Observe that, if (4.25) holds, then π_x is unique. Indeed, if (4.25) holds for two k-planes π_i $(i = 1, 2)$, by the uniqueness of the local weak* convergence of measures (see Remark 1.108), then

$$\mathcal{H}^k \llcorner \pi_1 = \mathcal{H}^k \llcorner \pi_2\,. \tag{4.28}$$

Since $\mathrm{spt}(\mathcal{H}^k\llcorner\pi_i) = \pi_i$ $(i = 1, 2)$, by (4.28), $\pi_1 = \pi_2$.

Definition 4.14 (Approximate tangent plane to a set)**.** Let $\Gamma\subset\mathbb{R}^n$ be such that $\mathcal{H}^k(\Gamma\cap K) < \infty$ for each compact set $K\subset\mathbb{R}^n$. A k-dimensional plane π_x is said to be *the approximate tangent plane to* Γ *at* x, if (4.25) holds. Then we denote $T_x\Gamma := \pi_x$.

Remark 4.15. We will see that the set of points $x\in\Gamma$ such that (4.25) holds true depends only on the Radon measure $\mu = \mathcal{H}^k\llcorner\Gamma$. It is a locally $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^n$ and is unchanged if we modify Γ on and by $\mathcal{H}^k$-null sets (see Theorem 4.21).

Lemma 4.16 (Approximate tangent plane to a regular Lipschitz image)**.** *Let* $f : \mathbb{R}^k \to \mathbb{R}^n$ *be a Lipschitz function and let* (f, E) *define a regular Lipschitz image in* $\mathbb{R}^n$*. If* $\Gamma = f(E)$*, then*

$$T_{f(z)}\Gamma = df(z)(\mathbb{R}^k)\ \mathcal{L}^k\textit{-a.e. } z\in E\,. \tag{4.29}$$

In particular

$$\exists\, T_x\Gamma = df(f^{-1}(x))(\mathbb{R}^k)\ \mathcal{H}^k\textit{-a.e. } x\in\Gamma\,. \tag{4.30}$$

Before the proof of Lemma 4.16, let us point out the following

Exercise: Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be differentiable at $z\in\mathbb{R}^k$ and assume that $\mathbf{J}f(z) > 0$. Then there exist λ, $s_0 > 0$ such that

$$|f(z') - f(z)| \geqslant \lambda\,|z'-z| \quad \forall\, z'\in B(z,s_0)\,. \tag{4.31}$$

(**Hint:** First observe that, since $\mathbf{J}f(z) > 0$, $\mathrm{rank}(df(z)) = k$. This implies that

$$\lambda_0 := \min\left\{|df(z)(v)| : v\in\mathbb{R}^k,\ |v| = 1\right\} > 0.$$

On the other hand, since f is differentiable at z, one can prove that, if $\lambda = \lambda_0/2$ there exits $s_0 > 0$ such that (4.31) holds.)

Proof. If $\varphi\in\mathbf{C}^0_c(\mathbb{R}^n)$, then by the integration with respect to a push- forward measure (see (1.69)) and the change of variables (see Theorem 3.56) we have

$$\begin{aligned}
\frac{1}{r^k}\int_{\mathbb{R}^n}\varphi\, d\left((\Phi_{x,r})_\#(\mathcal{H}^k\llcorner\Gamma)\right) &= \frac{1}{r^k}\int_\Gamma \varphi\circ\Phi_{x,r}\, d\mathcal{H}^k = \frac{1}{r^k}\int_\Gamma \varphi\left(\frac{y-x}{r}\right)\, d\mathcal{H}^k(y)\\
&= \frac{1}{r^k}\int_E \varphi\left(\frac{f(w)-f(z)}{r}\right)\mathbf{J}f(w)\, d\mathcal{L}^k(w)\\
&= \int_{\mathbb{R}^k} u_r(w)\, d\mathcal{L}^k(w)\,,
\end{aligned}$$

where $u_r : \mathbb{R}^k \to \mathbb{R}$ is defined as

$$u_r(w) := \chi_E(z+rw)\,\varphi\left(\frac{f(z+rw)-f(z)}{r}\right)\mathbf{J}f(z+rw)\,.$$

Let $(r_h)_h$ be an arbitary sequence of positive numbers satisfying $\lim_{h\to\infty} r_h = 0$. Let z be a Lebesgue point of χ_E and $\mathbf{J}f$, and f is differentiable at z, it is easy to see that, up to a subsequence, for each $R > 0$

$$\lim_{h\to\infty} u_{r_h}(w) = u_0(w) := \varphi\left(df(z)(w)\right)\mathbf{J}f(z) \quad \mathcal{L}^k\text{-a.e. } w \in B(R)\,.$$

and then

$$\lim_{h\to\infty} u_{r_h}(w) = u_0(w) \quad \mathcal{L}^k\text{-a.e. } w \in \mathbb{R}^k\,. \tag{4.32}$$

It is also easy to check that

$$|u_r(w)| \leqslant \sup_{\mathbb{R}^n}|\varphi|\,Lip(f)^k \quad \mathcal{L}^k\text{-a.e. } w \in \mathbb{R}^k,\ \forall\, r > 0\,. \tag{4.33}$$

We are going now to prove that there exist r_0 and $R_0 > 0$ such that

$$\operatorname{spt}(u_r) \subset B(R_0) \quad \forall\, r \in (0, r_0)\,. \tag{4.34}$$

Indeed, from (4.32)-(4.34), the dominated convergence theorem and the area formula, it will follow that

$$\begin{aligned}\lim_{h\to\infty}\frac{1}{r_h^k}\int_\Gamma \varphi\circ\Phi_{x,r_h}\,d\mathcal{H}^k &= \lim_{h\to\infty}\int_{\mathbb{R}^k} u_{r_h}(w)\,d\mathcal{L}^k(w) = \int_{\mathbb{R}^k} u_0(w)\,d\mathcal{L}^k(w)\\ &= \int_{\mathbb{R}^k}\varphi\left(df(z)(w)\right)\mathbf{J}f(z)\,d\mathcal{L}^k(w) = \int_{df(z)(\mathbb{R}^k)}\varphi\,d\mathcal{H}^k\end{aligned}$$

for each arbitrary sequence $(r_h)_h$ of positive real numbers with $\lim_{h\to\infty} r_h = 0$. Thus (4.29) is proved. Finally, let us show (4.34). By the previous exercise, there exist λ, $s_0 > 0$ such that

$$|f(z') - f(z)| \geqslant \lambda\,|z'-z| \quad \forall\, z' \in B(z, s_0)\,. \tag{4.35}$$

Moreover, if $R > 0$ is such that $\operatorname{spt}(\varphi) \subset B(R)$, then

$$|f(z+rw) - f(z)| \leqslant r\,R \quad \forall\, w \in \operatorname{spt}(u_r)\,. \tag{4.36}$$

By the compactness of E and injectivity of f on E one has

$$\inf\{|f(z') - f(z)| : z' \in E \setminus U(z, s_0)\} = \epsilon_0 > 0,$$

so that, by (4.35), one gets

$$|f(z') - f(z)| \geqslant \min\left\{\lambda, \frac{\epsilon_0}{d(E)}\right\}|z-z'| = c_0\,|z-z'| \quad \forall\, z' \in E\,. \tag{4.37}$$

In this way, if $w \in \operatorname{spt}(u_r)$ then $z + rw \in E$ and, by (4.36), $|f(z+rw) - f(z)| \leqslant Rr$, so that, by (4.37) $c_0\,r\,|w| \leqslant Rr$. This proves $\operatorname{spt}(u_r) \subset B(R_0)$ with $R_0 = R/c_0$, which shows (4.34). Eventually, Let $N \subset E$ denote the set of points $z \in E$ such that z is not a Lebesgue point of χ_E. Then $\mathcal{L}^k(N){=}0$. Thus, by the area formula

$$\mathcal{H}^k(f(N)) = 0\,.$$

Thus, by (4.29), for each $x \in \Gamma \setminus f(N)$, (4.32) holds. □

Proof of Theorem 4.12. **1st step.** We decompose $\Gamma = \Gamma_0 \cup_{h=1}^{\infty} f(E_h)$ as in Theorem 4.11. If we let $\Gamma_h = f_h(E_h)$, then by Lemma 4.16 we find that

$$\lim_{r\to 0^+} \frac{1}{r^k} \int_{\Gamma_h} \varphi \circ \Phi_{x,r}\, d\mathcal{H}^k = \int_{\pi_x} \varphi\, d\mathcal{H}^k \quad \forall\, \varphi \in \mathbf{C}_c^0(\mathbb{R}^n),\ \mathcal{H}^k - \text{-a.e. } x \in \Gamma_h\,, \tag{4.38}$$

where we have set $\pi_x := df_h(f_h^{-1}(x))(\mathbb{R}^k)$. Since $\mathcal{H}^k \llcorner \Gamma$ is a Radon measure on $\mathbb{R}^n$, by Corollary 3.38,

$$\Theta_k(\Gamma \setminus \Gamma_h, x) = \lim_{r\to 0^+} \frac{\mathcal{H}^k((\Gamma \setminus \Gamma_h) \cap B(x,r))}{\alpha_k^* \, r^k} = 0 \quad \mathcal{H}^k\text{-a.e. } x \in \Gamma_h\,. \tag{4.39}$$

For a given $\varphi \in \mathbf{C}_c^0(\mathbb{R}^n)$, suppose that $\operatorname{spt}(\varphi) \subset B(R_0)$. Then

$$\begin{aligned}
\frac{1}{r^k}\left|\int_{\Gamma\setminus\Gamma_h} \varphi\circ\Phi_{x,r}\, d\mathcal{H}^k\right| &= \frac{1}{r^k}\left|\int_{\Gamma\setminus\Gamma_h} \varphi\left(\frac{y-x}{r}\right)\, d\mathcal{H}^k(y)\right| \\
&= \frac{1}{r^k}\left|\int_{(\Gamma\setminus\Gamma_h)\cap B(x,R_0 r)} \varphi\left(\frac{y-x}{r}\right)\, d\mathcal{H}^k(y)\right| \\
&\leqslant \sup_{\mathbb{R}^n} |\varphi|\, \frac{\mathcal{H}^k\left((\Gamma\setminus\Gamma_h)\cap B(x,R_0 r)\right)}{r^k}\,.
\end{aligned}$$

Thus, by (4.39), it follows that

$$\begin{aligned}
&\lim_{r\to 0^+} \frac{1}{r^k}\int_{\Gamma\setminus\Gamma_h} \varphi\circ\Phi_{x,r}\, d\mathcal{H}^k \\
&= \lim_{r\to 0^+} \frac{1}{r^k}\int_{\Gamma\setminus\Gamma_h} \varphi\left(\frac{y-x}{r}\right)\, d\mathcal{H}^k(y) = 0 \quad \mathcal{H}^k\text{-a.e. } x \in \Gamma_h\,.
\end{aligned} \tag{4.40}$$

From (4.38) and (4.40), (4.26) follows.

2nd step. Let $x \in \Gamma$ satisfy (4.25). Since, by (4.5), $\mathcal{H}^k(\pi \cap \partial B_1) = 0$, by Theorem 1.113, we can infer that

$$\alpha_k^* = \mathcal{H}^k(\pi \cap B_1) = \lim_{r\to 0^+} \frac{\mathcal{H}^k \llcorner \Gamma\left(\Phi_{x,r}^{-1}(B_1)\right)}{r^k} = \lim_{r\to 0^+} \frac{\mathcal{H}^k(\Gamma \cap B(x,r))}{r^k}\,,$$

and thus (4.27) follows.

□

Let us also point out the following interesting locality result for the approximate tangent plane.

Proposition 4.17 (Locality of the approximate tangent plane)**.** *If* Γ_i *(*$i = 1,2$*) are locally* $\mathcal{H}^k$*-rectifiable sets of* $\mathbb{R}^n$*, then for* $\mathcal{H}^k$*-a.e.* $x \in \Gamma_1 \cap \Gamma_2$

$$T_x\Gamma_1 = T_x\Gamma_2\,.$$

Proof. See [Mag, Proposition 10.5]. □

Example 4.18 (Approximate tangent space to a Lipschitz $(n-1)$-graph)**.** If $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function, and we define $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ as $f(z) = (z, \phi(z))$, $z \in \mathbb{R}^{n-1}$, then, as pointed out in Example 4.6, $\Gamma := f(\mathbb{R}^{n-1}) = \operatorname{graph}(\phi)$ is locally $\mathcal{H}^{n-1}$-rectifiable and. Let us now show that for $\mathcal{L}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$, if $\nu(z) := (-\nabla\phi(z), 1)$,

$$T_{f(z)}\Gamma = \nu(z)^\perp := \{v \in \mathbb{R}^n : (v, \nu(z))_{\mathbb{R}^n} = 0\}\,, \tag{4.41}$$

or, equivalently, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$,

$$T_x\Gamma = \nu(f^{-1}(x))^{\perp} . \tag{4.42}$$

In particular this implies that the classical tangent space to a Lipschitz $(n-1)$-dimensional graph agrees $\mathcal{H}^{n-1}$-a.e. with the approximate tangent space.

Let us first observe that f is Lipschitz, injective and it actually satisfies

$$|z - z'| \leqslant |f(z) - f(z')| \leqslant \sqrt{1 + Lip(\phi)^2}\, |z - z'| \quad \forall\, z, z' \in \mathbb{R}^{n-1} . \tag{4.43}$$

Moreover, by (3.73),

$$\mathbf{J}f(z) = \sqrt{1 + |\nabla\phi(z)|^2} \quad \mathcal{L}^{n-1}\text{-a.e. } z \in \mathbb{R}^{n-1} \tag{4.44}$$

and, by the area formula,

$$\mathcal{H}^{n-1} \llcorner \Gamma(E) = \int_{f^{-1}(\Gamma\cap E)} \sqrt{1 + |\nabla\phi(z)|^2} d\mathcal{L}^{n-1} \tag{4.45}$$

for each Borel set $E \subset \mathbb{R}^n$.

Let $A_1 \subset \mathbb{R}^{n-1}$ denote the set of points x satysfying:

- ϕ is differentiable at x;
- x is a Lesbegue point of $\partial_j\phi$ for each $j = 1, \ldots, n-1$ (with respect to the $(n-1)$-dimensional Lebesgue measure $\mathcal{L}^{n-1}$);

and let $A_2 \subset A_1$ denote the set of points of density 1 in A_1 (with respect to the $(n-1)$-dimensional Lebesgue measure $\mathcal{L}^{n-1}$). By Rademacher's and Lebesgue point theorems (see Theorems 3.19 and 2.1) and the density of a set (see Corollary 2.18) A_i $(i = 1, 2)$ are measurable and they have full measure, that is $\mathcal{L}^{n-1}(\mathbb{R}^{n-1} \setminus A_i) = 0$ $(i = 1, 2)$. Let us define

$$A_R := A_2 \cap B(R) \text{ and } \Gamma_R := f(A_R) \text{ if } R > 0 .$$

Since A_R is measurable and $\mathcal{L}^{n-1}(A_R) < \infty$, by the approximation of Radon measures by means of compact sets (see Theorem 1.14), we can find and increasing sequence of compact sets $(E_h)_h$ such that

$$E_h \subset A_R \;\forall\, h \text{ and } \mathcal{L}^{n-1}(A_R \setminus \cup_{h=1}^{\infty} E_h) = 0 . \tag{4.46}$$

Let

$$\Gamma_{R,0} := f(A_R \setminus \cup_{h=1}^{\infty} E_h), \quad \Gamma_{R,h} := f(E_h) .$$

Then, by construction and (4.45),

$$\Gamma_R = \Gamma_{R,0} \cup (\cup_{h=1}^{\infty} \Gamma_{R,h}) \text{ and } \mathcal{H}^{n-1}(\Gamma_{R,0}) = 0 ;$$

by definiton of A_R, (4.43), (4.44), and (4.45)

$$\Gamma_{R,h} = f(E_h) \text{ is a regular Lipschitz image for each } h .$$

Applying Lemma 4.16, for a given h, it follows that

$$T_{f(z)}\Gamma_{R,h} = df(z)(\mathbb{R}^{n-1}) \quad \mathcal{L}^{n-1}\text{-a.e. } z \in E_h.$$

By standard linear algebra, we can infer that

$$df(z)(\mathbb{R}^{n-1}) = \text{span}\,\{v_1(z), \ldots, v_{n-1}(z)\} = \nu(z)^{\perp} \quad \mathcal{L}^{n-1}\text{-a.e. } z \in E_h ,$$

with

$$v_1(z) := e_1 + \partial_1\phi(z)\, e_n, \ldots, v_{n-1}(z) := e_{n-1} + \partial_{n-1}\phi(z)\, e_n ,$$

where $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{R}^n$. It is easy to see that vectors $v_i(z) \in \nu(z)^\perp$ for each $z \in E_h$, $i = 1, \dots, n-1$ and they are a basis of the subspace $\nu(z)^\perp$. Thus, we have proved that

$$T_{f(z)}\Gamma_{R,h} = \nu(z)^\perp \quad \mathcal{L}^{n-1}\text{-a.e. } z \in E_h.$$

or that is equivalent

$$T_x\Gamma_{R,h} = \nu(f^{-1}(x))^\perp \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma_{R,h}\,. \tag{4.47}$$

By the locality of the approximate tangent plane (see Proposition 4.17) and (4.47), it follows that, for every h,

$$T_x\Gamma = T_x\Gamma_{R,h} = \nu(f^{-1}(x))^\perp \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma_{R,h} = \Gamma_{R,h} \cap \Gamma\,,$$

and then

$$T_x\Gamma = \nu(f^{-1}(x))^\perp \quad \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma_R = \Gamma_R \cap \Gamma\,. \tag{4.48}$$

By (4.48), since $R > 0$ is arbitrary, (4.42) follows. Finally, by (4.45), (4.42) and (4.41) are equivalent.

Let us point out the striking fact that (4.27) implies the rectifiability. Indeed

Theorem 4.19 (Besicovitch-Marstrand-Mattila). *Let E a Borel set with $\mathcal{H}^k(E) < \infty$. Then the following are equivalent:*

(i) *E is $\mathcal{H}^k$-rectifiable;*
(ii) *there exists $\Theta_k(E, x) = 1$ for $\mathcal{H}^k$-a.e. $x \in E$.*

Proof. Implication (i)⇒(ii) follows from Theorem 4.12. Implication (ii)⇒(ii) is much harder and can be found in [Ma, Theorem 17.6]. □

Remark 4.20. Preiss improved these results in [Pre2] proving that the existence of $\Theta_k(E, x) \in (0, \infty)$ implies the $\mathcal{H}^k$-rectifiabilty of E.

4.3. Blow-ups of Radon measures on $\mathbb{R}^n$ and rectifiability.

We now prove a converse statement to Theorem 4.12, which plays an important role in GMT and, in particular, when studying the structure of sets of finite perimeter (see Chap. V).

Theorem 4.21 (Rectifiability by convergence of the blow-ups). *If μ is a Radon measure on $\mathbb{R}^n$, Γ is a Borel set in $\mathbb{R}^n$, μ is concentrated on Γ (that is $\mu = \mu \llcorner \Gamma$), and, for every $x \in \Gamma$, there exists a k-dimensional plane π_x in $\mathbb{R}^n$ such that*

$$\frac{(\Phi_{x,r})_\# \mu}{r^k} \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x \ \text{as } r \to 0^+,$$

then $\mu = \mathcal{H}^k \llcorner \Gamma$ and Γ is locally $\mathcal{H}^k$-rectifiable.

The proof of Theorem 4.21 relies on a simple criterion for $\mathcal{H}^k$ rectifiability, very useful in GMT.

Definition 4.22 (Cones). If π is a k-plane, define the *cone* $K(\pi, t)$ and opening $t > 0$, as

$$K(\pi,t) := \left\{y \in \mathbb{R}^n : |P_\pi^\perp(y)| \leqslant t\,|P_\pi(y)|\right\} = \left\{y \in \mathbb{R}^n : |y| \leqslant \sqrt{1+t^2}\,|P_\pi(y)|\right\} = \left\{y \in \mathbb{R}^n : d(y,\pi) \leqslant \sqrt{\frac{1}{1+t^2}}\,|y|\right\} \tag{4.49}$$

Notice that $K(\pi, t)$ is invariant by dilations, that is $s\,K(\pi, t) = K(\pi, t)$ for each $s > 0$, $K(\pi, t)$ reduces to π if $t = 0$, and $K(\pi, t) \setminus \{0\} \uparrow \mathbb{R}^n \setminus \pi^\perp$ as $t \uparrow \infty$.

Subsets of Lipschitz k-graphs can be easily characterised by cones. In fact

Exercise: Let $S \subset \mathbb{R}^n$. Then the following are equivalent:

(i) there exist a k-dimensional plane π and an opening $t > 0$ such that $S \subset x + K(\pi, t)$ for each $x \in S$;

(ii) there exist a k-dimensional plane π , a unique function $\phi : E \subset \pi \to \pi^\perp \subset \mathbb{R}^n$ and a constant $t > 0$ such that ϕ is t-Lipschitz and $\mathrm{graph}(\phi) := \{z + \phi(z) : z \in E\} = S$.

(**Hint:** Implication (ii)⇒ (i) is trivial. Implication (i)⇒ (ii) follows by noticing that:

a) if x_1, $x_2 \in S$ and $P_\pi(x_1) = P_\pi(x_2)$ then $x_1 = x_2$;

b) for each $z \in E := P_\pi(S) \subset \pi$ there exists a unique $y := \phi(z) \in \pi^\perp$ such that $z + y \in S$ and $\phi : E \subset \pi \to \pi^\perp \subset \mathbb{R}^n$ is a t-Lipschitz function.)

Theorem 4.23 (Rectifiability criterion). *If $\Gamma \subset \mathbb{R}^n$ is a compact set, π is a k-dimensional plane in $\mathbb{R}^n$, and there exist δ and t positive with*

$$\Gamma \cap B(x,\delta) \subset x + K(\pi,t) \quad \forall\, x \in \Gamma\,, \tag{4.50}$$

then Γ is $\mathcal{H}^k$-rectifiable, since there exist finitely many Lipschitz maps $f_h : \mathbb{R}^k \to \mathbb{R}^n$ $(h = 1, \dots, N)$ and compact sets $F_h \subset \mathbb{R}^k$ with

$$\Gamma = \cup_{h=1}^N f_h(F_h)\,.$$

Proof. Let $z \in \Gamma$, then notice that

$$\Gamma \cap B(z,\delta/2) \subset x + K(\pi,t) \quad \forall\, x \in \Gamma \cap B(z,\delta/2)\,. \tag{4.51}$$

Since Γ is compact, there exist $x_1, \dots, x_N \in \Gamma$ such that $\Gamma \subset \cup_{h=1}^N B(x_h, \delta/2)$ and (4.51) holds with $z = x_h$ $h = 1, \dots, N$. By the previous exercise, for each $h = 1, \dots, N$ there exists a t-Lipschitz function

$$\phi_h : \tilde F_h := P_\pi\left(\Gamma \cap B(x_h,\delta/2)\right) \subset \pi \to \pi^\perp \subset \mathbb{R}^n$$

such that

$$\mathrm{graph}(\phi_h) = \Gamma \cap B(x_h,\delta/2)\,, \tag{4.52}$$

and

$$\Gamma = \cup_{h=1}^N \left(\Gamma \cap B(x_h,\delta/2)\right) = \cup_{h=1}^N \mathrm{graph}(\phi_h)\,,$$

with $\tilde F_h \subset \pi$ $(h = 1, \dots, N)$ compact sets. Let us define, for $h = 1, \dots, N$,

$$g_h : \tilde F_h \subset \pi \to \mathbb{R}^n, \quad g_h(z) := z + \phi_h(z)\,.$$

Then g_h is Lipschitz and, by Corollary 3.17, we can assume that g_h is defined on the whole π. Let us now define

$$f_h := g_h \circ I_\pi : \mathbb{R}^k \to \mathbb{R}^n$$

where I_π is the map defined in (4.1), we get the desired conclusion by choosing $F_h := I_\pi^{-1}(\tilde{F}_h)$ $(h = 1, \ldots, N)$. □

Remark 4.24. As a consequence of (4.52), we also proved that a compact set $\Gamma \subset \mathbb{R}^n$ satisfying (4.50) is the union of finitely many Lipschitz k-graphs, according to Example 4.6 and then a $\mathcal{H}^k$- rectifiable set.

We will also need a technical lemma concerning inclusions between cones and balls.

Lemma 4.25. *There is $s_0 \in (0,1)$ such that for each two k-dimensional planes π and σ satisfying $|\pi - \sigma|_G < s_0$, it holds that*

$$B(w, s_0|w|) \cap K(\pi, 1) = \emptyset \quad \forall\, w \in \mathbb{R}^n \setminus K(\sigma, 2)\,. \tag{4.53}$$

Proof. Let us begin to observe that, since sets $U(w, s_0|w|)$, $K(\pi, 1)$ and $\mathbb{R}^n \setminus K(\sigma, 2)$ are invariant by dilations, without loss of generality, we can prove (4.53) for each $w \in \mathbf{S}^{n-1} \cap (\mathbb{R}^n \setminus K(\sigma, 2))$. By contradiction, assume there are a sequence $(s_h)_h \downarrow 0$, two sequences of k-dimensional planes $(\pi_h)_h$ and $(\sigma_h)_h$ and two sequences $(w_h)_h$ and $(z_h)_h$ in $\mathbb{R}^n$ satisfying, for each h,

$$|\pi_h - \sigma_h|_G < s_h\,, \tag{4.54}$$

$$z_h \in B(w_h, s_h) \cap K(\pi_h, 1) \iff |w_h - z_h| \leqslant s_h \text{ and } |z_h| \leqslant \sqrt{2}\,|P_{\pi_h}(z_h)|\,, \tag{4.55}$$

$$w_h \in \mathbf{S}^{n-1} \cap (\mathbb{R}^n \setminus K(\sigma_h, 2)) \iff |w_h| = 1 \text{ and } |w_h| > \sqrt{5}\,|P_{\sigma_h}(w_h)|\,. \tag{4.56}$$

By compactness (recall (4.3)), we can assume that there exist two k-dimensional planes π^* and σ^*, a pont $w^* \in \mathbf{S}^{n-1}$ such that

$$\lim_{h\to\infty} |\pi_h - \pi^*|_G = \lim_{h\to\infty} |\sigma_h - \sigma^*|_G = 0\,, \tag{4.57}$$

$$\lim_{h\to\infty} |w - w^*| = 0\,. \tag{4.58}$$

Thus, by (4.54)-(4.58), we can infer that

$$\pi^* = \sigma^* \text{ and } \exists \lim_{h\to\infty} z_h = w^*\,. \tag{4.59}$$

By the definition of distance $|\cdot|_G$ and (4.60), the sequences of operators $(P_{\pi_h})_h$ and $(P_{\sigma_h})_h$ uniformly converge on $\mathbf{S}^{n-1}$ to P_{π^*}. In particular we also get that

$$\lim_{h\to\infty} |P_{\pi_h}(z_h) - P_{\pi^*}(w^*)| = \lim_{h\to\infty} |P_{\sigma_h}(w_h) - P_{\pi^*}(w^*)| = 0\,. \tag{4.60}$$

By (4.60), we can pass to the limit as $h \to \infty$ in the two inequalities in (4.55) and (4.57) and we get

$$\sqrt{5}\,|P_{\pi^*}(w^*)| \leqslant 1 \leqslant \sqrt{2}\,|P_{\pi^*}(w^*)|\,,$$

and then a contradiction. □

An other useful tool in GMT is the following version of Severini-Egoroff's theorem 1.18 which applies to Radon measures on $\mathbb{R}^n$ and a family of convergent functions rather than a sequence of convergent functions.

Lemma 4.26 (Severini-Egoroff's theorem for a convergent family of functions)**.** *Let φ be a finite Radon outer measures on $\mathbb{R}^n$, let $\Gamma \subset \mathbb{R}^n$ be φ-measurable and let $f_r : \Gamma \to \mathbb{R}$ ($r \geqslant 0$) be a family of functions such that*

$$\exists \lim_{r\to 0^+} f_r(x) = f_0(x) \quad \varphi\text{-a.e. } x \in \Gamma\,.$$

Then, for each $\epsilon > 0$, there exists a compact set $\Gamma' \subset \Gamma$ such that

$$\varphi(\Gamma \setminus \Gamma') < \epsilon\,,$$

and

$$f_r \to f_0 \text{ uniformly on } \Gamma' \text{ as } r \to 0^+, \text{ i.e. } \lim_{r\to 0^+} \sup_{x\in\Gamma'} |f_r(x) - f(x)| = 0\,.$$

Proof. Let us define the sequence of functions $g_h : \Gamma \to \mathbb{R}$

$$g_h(x) := \sup\{|f_r(x) - f(x)| : 0 < r < 1/h\} \quad h = 1, 2, \ldots\,.$$

Observe that g_h ($h = 1, 2, \ldots$) is well- defined, φ- measurable and

$$\lim_{h\to\infty} g_h(x) = 0 \quad \mu\text{-a.e. } x \in \Gamma\,.$$

Indeed, it is easy to see that

$$g_h(x) := \sup\{|f_r(x) - f(x)| : r \in \mathbb{Q},\ 0 < r < 1/h\}\,.$$

thus, since the family $\{f_r : r \geqslant 0\}$ is φ-measurable, also g_h is φ-measurable. Given $\epsilon > 0$, applying Severini-Egoroff's theorem 1.18 to sequence $(g_h)_h$ we get the existsnce of a φ-measurable set $E \subset \Gamma$ such that

$$\varphi(\Gamma \setminus E) < \epsilon/2 \text{ and } g_h \to 0 \text{ uniformly on } E \text{ as } h \to \infty\,. \tag{4.61}$$

On the other hand, by the approximation theorem for Radon measures (see Theorem 1.14), there is a compact set $\Gamma' \subset E$ such that

$$\varphi(E \setminus \Gamma') < \epsilon/2\,. \tag{4.62}$$

Therefore, by (4.61) and (4.62), we get the desired conclusion. □

Proof of Theorem 4.21. Let s_0 be the positive constant in Lemma 4.25. Then, by (4.3), there is finite family of k-dimensional planes $\sigma_1, \ldots, \sigma_N$ such that

$$\min_{1\leqslant h\leqslant N} |\sigma_h - \pi|_G \leqslant \frac{s_0}{2} \quad \text{for each } k\text{-dimensional plane } \pi\,. \tag{4.63}$$

Let us now divide the proof in three steps.

1st step. We show that if Γ' is a compact subset of Γ and assume that the limit relations

$$\lim_{r\to 0^+} \frac{\mu(B(x,r))}{\alpha_k^*\, r^k} = 1\,,$$

$$\lim_{r\to 0^+} \frac{\mu\left(B(x,r) \setminus (x + K(\pi_x, 1))\right)}{\alpha_k^*\, r^k} = 0\,,$$

hold *uniformly* with respect to $x \in \Gamma'$, then Γ' is $\mathcal{H}^k$-rectifiable. Thus, assume that: for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in \Gamma'$, $r \in (0, \delta)$

$$1 - \epsilon < \frac{\mu(B(x,r))}{\alpha_k^*\, r^k} < 1 + \epsilon\,, \tag{4.64}$$

$$\mu\left(B(x,r)\setminus(x+K(\pi_x,1))\right) < \epsilon\,\alpha_k^*\,r^k\,. \tag{4.65}$$

Let

$$\Gamma'_h := \left\{x\in\Gamma' : |\sigma_h-\pi_x|_G \leqslant \frac{s_0}{2}\right\} \quad 1\leqslant h\leqslant N\,.$$

Then let us show that

$$\Gamma'_h\cap B(x,\delta)\subset x+K(\sigma_h,2) \quad \forall\, x\in\Gamma'_h,\ h=1,\dots,N\,. \tag{4.66}$$

By contradiction, if $x\in\Gamma'_h$ and $y\in B(x,\delta)\cap\Gamma'_h\setminus\{x\}$ but $y\notin(x+K(\sigma_h,2))$, that is $y-x\in\mathbb{R}^n\setminus K(\sigma_h,2)$, then, by (4.53),

$$B(y,s_0|y-x|)\subset\mathbb{R}^n\setminus(x+K(\pi_x,1))\,. \tag{4.67}$$

Since $s_0\in(0,1)$

$$B(y,s_0|y-x|)\subset B(x,2\,|y-x|)$$

by (4.67), it follows that

$$B(y,s_0|y-x|)\subset B(x,2\,|y-x|)\setminus(x+K(\pi_x,1))\,.$$

Applying (4.65) (at x with $r=2|y-x|$) and (4.64) (at y with $r=s_0|y-x|$)

$$\epsilon\,\alpha_k^*\,2^k\,|y-x|^k \geqslant (1-\epsilon)\,\alpha_k^*\,s_0^k\,|y-x|^k$$

a contradiction, as soon as ϵ is small enough with respect to k and s_0. This proves (4.66). By the rectifiability criterion (Theorem 4.23) , Γ'_h is thus $\mathcal{H}^k$-rectifiable for $h=1,\dots,N$. By (4.63), $\Gamma'\subset\cup_{h=1}^N\Gamma'_h$. Thus Γ' is $\mathcal{H}^k$ -rectifiable.

2nd step. Let us prove that Γ is countable $\mathcal{H}^k$-rectifiable. We have

$$\lim_{r\to0^+}\frac{\mu(B(x,r))}{\alpha_k^*\,r^k} = 1\,, \tag{4.68}$$

$$\lim_{r\to0^+}\frac{\mu\left(B(x,r)\setminus(x+K(\pi_x,1))\right)}{\alpha_k^*\,r^k} = 0 \tag{4.69}$$

for each $x\in\Gamma$. Indeed, since, by (4.5), $\mathcal{H}^k\llcorner\pi(\partial B_1) = \mathcal{H}^k(\pi\cap\partial B_1) = 0$, therefore, by Theorem 1.113,

$$\alpha_k^* = \mathcal{H}^k(\pi_x\cap B_1) = \lim_{r\to0^+}\frac{(\Phi_{x,r})_\#\mu(B(x,r))}{r^k} = \lim_{r\to0^+}\frac{\mu((B(x,r))}{r^k}\,,$$

that is (4.68). Let us now check (4.69) in a similar way. Notice that, if $E := B_1\setminus K(\pi_x,1)$, then

$$x+r\,E = B(x,r)\setminus(x+K(\pi_x,1))\ ,$$

and

$$\pi_x\cap E=\emptyset,\ \pi_x\cap\partial E=\{0\}\,.$$

Thus, since $\mathcal{H}^k\llcorner\pi(\partial E) = 0$, by (4.24),

$$0 = \mathcal{H}^k(\pi\cap E) = \lim_{r\to0^+}\frac{(\Phi_{x,r})_\#\mu(E)}{r^k} = \lim_{r\to0^+}\frac{\mu\left(B(x,r)\setminus(x+K(\pi_x,1))\right)}{\alpha_k^*\,r^k}\,.$$

Let us now prove tha, for each Borel set $E\subset\mathbb{R}^n$

$$\mathcal{H}^k\llcorner\Gamma(E)\leqslant\mu(E)=\mu\llcorner\Gamma(E)\leqslant 2^k\,\mathcal{H}\llcorner\Gamma(E)\,. \tag{4.70}$$

Indeed, by the estimates of the upper density of a Radon measure (see Theorem 3.37) since, by (4.68),

$$1 \leqslant \Theta_k(\mu, x) = \Theta_k^*(\mu, x) \leqslant 1 \quad \forall\, x \in \Gamma\,,$$

(4.70) follows.

Given $R > 0$, since $\mu(B(R)) < \infty$, by applying Lemma 4.26 with $\varphi = \mu \llcorner B(R)$, we can infer the existence of a compact set $\Gamma_1 \subset \Gamma$ such that the limit relations (4.68) and (4.69) hold uniformly on Γ_1 with

$$\mu((\Gamma \cap B(R)) \setminus \Gamma_1) < \frac{\mu(\Gamma \cap B(R))}{2}\,.$$

By an induction procedure, we can construct a disjoint sequence of compact sets $(\Gamma_h)_h$ such that

$$\mu\left((\Gamma \cap B(R) \setminus (\cup_{h=1}^m \Gamma_h)\right) < \frac{\mu(\Gamma \cap B(R))}{2^m} \quad \forall\, m = 1, 2, \ldots\,. \tag{4.71}$$

(4.72) the limit relations (4.68) and (4.69) hold uniformly on Γ_h

By step 1 and (4.72), it follows that Γ_h $(h = 1, 2, \ldots,)$ is $\mathcal{H}^k$- rectifiable. Since, by (4.71), $\mu((\Gamma \cap B(R)) \setminus (\cup_{h=1}^\infty \Gamma_h)) = 0$, by (4.70) we can also infer that $\mathcal{H}^k((\Gamma \cap B(R)) \setminus (\cup_{h=1}^\infty \Gamma_h)) = 0$. Thus, by definition,

$$\Gamma \cap B(R) \text{ is countably } \mathcal{H}^k\text{- rectifiable for each } R > 0\,,$$

and since $\Gamma = \cup_{h=1}^\infty (\Gamma \cap B(h))$, we also get that

$$\Gamma \text{ is countably } \mathcal{H}^k\text{- rectifiable}\,.$$

Let us now prove that Γ is actually locally $\mathcal{H}^k$- rectifiable. By (4.70), it follows that, since μ is a Radon o.m., $\mathcal{H}^k(\Gamma \cap K) < \infty$ for each compact set $K \subset \mathbb{R}^n$. Therefore we get the desired conclusion.

3rd step. Let us prove that $\mu = \mathcal{H}^k \llcorner \Gamma$ on $\mathcal{P}(\mathbb{R}^n)$. Since, by (4.70), $\mathcal{H}^k \llcorner \Gamma$ is a Radon o.m., absolutely continuous with respect to μ, by the differentiation theorem for positive Radon measures (see Theorem 2.15), $\mathcal{H}^k \llcorner \Gamma = w\, d\mu$ with

$$w(x) = D_{\mathcal{H}^k \llcorner \Gamma}\mu(x) = \lim_{r \to 0^+} \frac{\mu(B(x, r))}{\mathcal{H}^k(\Gamma \cap B(x, r))} \quad \mathcal{H}^k\text{-a.e.} x \in \Gamma\,. \tag{4.73}$$

By (4.26), (4.68) and (4.73), it follows that $w(x) = 1$ for $\mathcal{H}^k$-a.e. $x \in \Gamma$. Thus $\mathcal{H}^k \llcorner \Gamma = \mu$ on the class of Borel sets. By Remark 1.15, it follows that they agree on $\mathcal{P}(\mathbb{R}^n)$. □

Purely unrectifiable sets

Definition 4.27 (Purely unrectifiable sets)**.** Let $E \subset \mathbb{R}^n$ be a Borel set. We say that E is *purely $\mathcal{H}^k$-unrectifiable* if $\mathcal{H}^k(E \cap f(\mathbb{R}^k)) = 0$ for any Lipschitz function $f : \mathbb{R}^k \to \mathbb{R}^n$.

Equivalently, we might say that E is purely $\mathcal{H}^k$-unrectifiable if $\mathcal{H}^k(E \cap F) = 0$ for any countably $\mathcal{H}^k$-rectifiable set F. There exist examples of purely $\mathcal{H}^k$-unrectifiable with Hausdorff dimension strictly greater than k (for $k = 1$ one example is the *von Koch snowflake* in the plane, whose Hausdorff dimension is $\log 4/\log 3$, see [Ma, 4.13] and [Fa, Introduction]). In the next example we show how a purely $\mathcal{H}^1$- unrectifiable set in $\mathbb{R}^2$ with Hausdorff dimension 1 can be constructed.

Example 4.28 (An unrectifiable set)**.** Let $C := C(1/4)$ be the Cantor set defined in (3.51) with $\lambda = 1/4$ and let $E := C \times C \subset \mathbb{R}^2$ (called the *Cantor dust* in the unit square). It is easy to see that E still presents a self-similar structure. Indeed, let

$$C_k := C_k(1/4) = \cup_{j=1}^{2^k} I_{k,j} \quad k = 1, 2, \dots ,$$

be the sets defined in (3.51) for $C := C(1/4)$'s definition. Since, by definition,

$$C := \cap_{k=1}^{\infty} C_k \, ,$$

then

$$E := C \times C = \cap_{h=1}^{\infty} (C_k \times C_k) \, .$$

with

$$\begin{aligned} C_k \times C_k &= \left(\cup_{j=1}^{2^k} I_{k,j} \right) \times \left(\cup_{l=1}^{2^k} I_{k,l} \right) \\ &= \cup_{i,l=1}^{2^k} \left(I_{k,j} \times I_{k,l} \right) = \cup_{i,l=1}^{2^k} Q_{k,jl} \end{aligned} \tag{4.74}$$

where $Q_{k,jl}$ j, $l = 1, \dots, 2^k$, is a family of 4^k closed disjoint squares with side length 4^{-k} for $k = 1, 2, \dots$.

Then we claim that:

(i) $0 < 3/\sqrt{5} \leqslant \mathcal{H}^1(E) \leqslant \sqrt{2}$.

(ii) $\Theta^1_*(E, (x,y)) := \liminf_{r \to 0^+} \dfrac{\mathcal{H}^1(E \cap B((x,y), r)}{2\, r} \leqslant \dfrac{1}{2} < 1$ for each $(x, y) \in E$.

(iii) Let P_h denote the union of the boundaries of 4^h squares $Q_{h,jl}$ for $j, l = 1, \dots, 2^h$. Then it holds that $(\mathcal{H}^1 \llcorner P_h)_h$ weakly* converges to $c\,\mathcal{H}^1 \llcorner E$ on $\mathcal{B}(\mathbb{R}^2)$ as $h \to \infty$ with $c = 4/\mathcal{H}^1(E)$.

From (i), it follows that $\mathrm{Hdim}(E) = 1$ and, by Theorem 4.12 and (ii), that E is purely $\mathcal{H}^1$-unrectifiable. Claim (iii) shows that weak* convergence is an approximation too weak to preserve the concept of rectifiability. Indeed sets P_h are finite unions of Lipschitz curves and then $\mathcal{H}^1$-rectifiable, instead of E, which is purely $\mathcal{H}^1$-unrectifiable. Finally notice that E is totally disconnected, that is its connected components are points. Indeed it can be proved that, if $\Gamma \subset \mathbb{R}^n$ is closed and connected with $\mathcal{H}^1(\Gamma) < \infty$, then Γ is $\mathcal{H}^1$-rectifiable (see Theorem 4.31).

Let us now prove (i),(ii) and (iii).

Proof of (i): From (4.99), it follows that, if $\delta_k := \sqrt{2}\, 4^{-k}$,

$$\mathcal{H}^1_{\delta_k}(E) \leqslant \sqrt{2} \quad \forall\, k$$

and so inequality $\mathcal{H}^1(E) \leqslant \sqrt{2}$ follows. We have now to prove that

$$\frac{3}{\sqrt{5}} \leqslant \mathcal{H}^1(E) \, . \tag{4.75}$$

Let

$$\pi := \{(x,y) \in \mathbb{R}^2 : y = 2x\}$$

and let $P_\pi : \mathbb{R}^2 \to \mathbb{R}^2$ denote the orthogonal projection of $\mathbb{R}^2$ on straight line π. If T is the segment of π defined by

$$T := \left\{(x,y) \in \mathbb{R}^2 : y = 2x,\, 0 \leqslant x \leqslant \frac{3}{5}\right\},$$

then T turns out to be the orthogonal projection on π both of square $Q_{0,1}$ and strip

$$S_{0,1} := \left\{(x,y) \in \mathbb{R}^2 : -\frac{x}{2} \leqslant y \leqslant -\frac{x}{2} + \frac{3}{2}\right\}$$

that is

$$P_\pi(S_{0,1}) = P_\pi(Q_{0,1}) = T\,, \tag{4.76}$$

which is quite evident since the orthogonal subspace to π turns out to be

$$\pi^\perp = \left\{(x,y) : y = -\frac{x}{2}\right\}.$$

We also claim that

$$P_\pi(E) = T\,. \tag{4.77}$$

Since

$$\mathcal{H}^1(T) = \frac{3}{\sqrt{5}}$$

and $P_\pi : \mathbb{R}^2 \to \mathbb{R}^2$ is a 1- Lipschitz map, by (4.77) and Propositon 3.22,

$$\frac{3}{\sqrt{5}} = \mathcal{H}^1(T) = \mathcal{H}^1(P_\pi(E)) \leqslant \mathcal{H}^1(E)\,,$$

and (4.75) follows. Thus we have only to show (4.77).

Thus it suffices to prove that

$$P_\pi(C_k \times C_k) = T \quad \text{for each } k = 1, 2, \dots, \tag{4.78}$$

in order to show (4.77). Let us define, for $k = 1, 2, \dots,$ the family of 4^k substrips of $S_{0,1}$

$$S_{k,h} := \left\{(x,y) \in \mathbb{R}^2 : -\frac{x}{2} + \frac{h-1}{4^k}\frac{3}{2} \leqslant y \leqslant -\frac{x}{2} + \frac{h}{4^k}\frac{3}{2}\right\} \quad h = 1, 2, \dots, 4^k\,,$$

It is easy to prove that,

$$S_{0,1} = \cup_{h=1}^{4^k} S_{k,h} \text{ for each } k = 1, 2, \dots, \tag{4.79}$$

and, for a given $k = 1, 2, \dots,$ for each integer $1 \leqslant h \leqslant 4^k$ there are unique integers $1 \leqslant j_h \leqslant 2^k$ and $1 \leqslant l_h \leqslant 2^k$ such that

$$Q_{k,h} := Q_{k,j_h l_h} \subset S_{k,h} \text{ and } P_\pi(Q_{k,h}) = P_\pi(S_{k,h})\,. \tag{4.80}$$

Thus we can relabelling the family of 4^k squares $Q_{k,jl}$ $j,l = 1,\dots,2^k$ by means of the family $Q_{k,h}$, $h = 1, 2, \dots, 4^k$, such that

$$C_k \times C_k = \cup_{h=1}^{4^k} Q_{k,h}\,, \tag{4.81}$$

$$d(Q_{k,h}, Q_{k,m}) \geqslant \frac{1}{2\,4^{k-1}} \text{ if } h \neq m\,. \tag{4.82}$$

By (4.87), (4.79), (4.80) and (4.81), we can infer that

$$\begin{aligned} P_\pi(C_k \times C_k) = &\; P_\pi\left(\cup_{h=1}^{4^k} Q_{k,h}\right) = \cup_{h=1}^{4^k} P_\pi(Q_{k,h}) \\ = &\; \cup_{h=1}^{4^k} P_\pi(S_{k,h}) = P_\pi\left(\cup_{h=1}^{4^k} S_{k,h}\right) \\ = &\; P_\pi(S_{0,1}) = T \end{aligned}$$

and (4.78) follows.

Proof of (ii): Let us begin to point out the self-similar structure of E . Let $\lambda = 1/4$ and define the following four similarities $S_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2, 3, 4)$

$$S_1(x, y) := (\lambda\, x, \lambda\, y), \quad S_2(x, y) := S_1(x, y) + (1 - \lambda, 0)$$
$$S_3(x, y) := S_1(x, y) + (0, 1 - \lambda), \quad S_4(x, y) := S_1(x, y) + (1 - \lambda, 1 - \lambda)\,.$$

Since

$$S_i(Q_{0,1}) = Q_{1,i} \text{ for } i = 1, 2, 3, 4\,, \tag{4.83}$$

it is easy to check that

$$\cup_{i=1}^4 S_i(Q_{0,1}) = \cup_{i=1}^4 Q_{1,i} = C_1 \times C_1 \tag{4.84}$$

and

$$\cup_{i=1}^4 S_i(C_k \times C_k) = C_{k+1} \times C_{k+1} \text{ for each } k \geqslant 1\,. \tag{4.85}$$

In particular it follows that

$$\cup_{i=1}^4 S_i(E) = E\,, \tag{4.86}$$

and

$$S_i(E) = E \cap Q_{1,i} \text{ for } i = 1, 2, 3, 4\,. \tag{4.87}$$

Moreover, as usual in a self-similar uniform structure, we can infer that

$$\mathcal{H}^1(E \cap Q_{k,i}) = \frac{1}{4^k}\,\mathcal{H}^1(E) \text{ for } k \geqslant 1,\; i = 1, 2, \ldots, 4^k\,. \tag{4.88}$$

Indeed, by induction on k, (4.88) holds for $k = 1$ by (4.83) and (4.86). Assume that (4.88) for k, then let us prove it hold for $k + 1$. Fix an integer $1 \leqslant s \leqslant 4^{k+1}$. Then it is clear that there exist integers $1 \leqslant i^* \leqslant 4$ and $1 \leqslant s^* \leqslant 4^k$ such that

$$Q_{k+1,h} = S_{i^*}(Q_{k,s^*})\;.$$

Therefore, by (4.83), (3.24), (3.25) and the inductive hypothesis,

$$\begin{aligned} \mathcal{H}^1(E \cap Q_{k+1,s}) = &\; \mathcal{H}^1\left(E \cap S_{i^*}(Q_{k,s^*})\right) = \mathcal{H}^1\left(E \cap Q_{1,i^*} \cap S_{i^*}(Q_{k,s^*})\right) \\ = &\; \mathcal{H}^1\left(S_{i^*}(E) \cap S_{i^*}(Q_{k,s^*})\right) = \frac{1}{4}\mathcal{H}^1\left(E \cap Q_{k,s^*}\right) = \frac{1}{4^{k+1}}\mathcal{H}^1(E)\;. \end{aligned}$$

Fix $(x, y) \in E$. By definition, $(x, y) \in C_k \times C_k$ for each $k = 1, 2, \ldots$, then there exists a unique integer $\bar{s} = \bar{s}(x, y) \in \{1, 2, \ldots, 4^k\}$ such that

$$(x, y) \in Q_{k,s} \text{ if and only if } h = \bar{s}\,, \tag{4.89}$$

and, if $r_k := \sqrt{2}\,4^{-k}$,

(4.90) $$E \cap Q_{k,\bar{s}} = E \cap B((x,y),r_k) \quad \forall\, k = 1,2,\dots.$$

Indeed because of (4.81) and (4.82), the family of cubes $Q_{k,s}$ $s = 1,2,\dots,4^k$ covering $C_k \times C_k$ are disjoint, which yields (4.89). Since $d(Q_{k,\bar{s}}) \leqslant r_k$, it is immediate that

$$E \cap Q_{k,\bar{s}} \subseteq E \cap B((x,y),r_k)\,.$$

The reverse inclusion follows by noticing that

$$B((x,y),r_k) \cap Q_{k,s} = \emptyset \text{ for each } s \neq \bar{s}\,,$$

otherwise,

$$d(Q_{k,\bar{s}}, Q_{k,s}) \leqslant r_k = \frac{\sqrt{2}}{4^k} < \frac{1}{2\,4^{k-1}} \text{ with } s \neq \bar{s}\,,$$

which contradicts (4.82). This proves (4.90). By (4.90), (4.88) and the previous claim (i), it follows that

$$\frac{\mathcal{H}^1(E \cap B((x,y),r_k)}{2\,r_k} = \frac{\mathcal{H}^1(E \cap Q_{k,\bar{s}})}{2\,r_k} \leqslant \frac{\mathcal{H}^1(E)}{2\sqrt{2}} \leqslant \frac{1}{2} \quad \forall\, k\,,$$

which implies

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^1(E \cap B((x,y),r)}{2\,r} \leqslant \liminf_{k\to\infty} \frac{\mathcal{H}^1(E \cap B((x,y),r_k)}{2\,r_k} \leqslant \frac{1}{2}\,,$$

and the proof is accomplished.

Proof of (iii): Let $\mu_h := \mathcal{H}^1 \llcorner P_h : \mathcal{B}(\mathbb{R}^2) \to [0,\infty]$ and $\lambda := c\,\mathcal{H}^1 \llcorner E : \mathcal{B}(\mathbb{R}^2) \to [0,\infty]$. Then, since

$$\mu_h(\mathbb{R}^2) = 4 \quad \forall\, h\,,$$

by the compactness of weak* convergence (see Theorem 1.121)), it is not restrictive to assume that there exists a Radon measure μ on $\mathbb{R}^2$ such that

(4.91) $$(\mu_h)_h \text{ weakly* converges to } \mu\,.$$

In particular we also get that $\mu_h \overset{*}{\rightharpoonup} \mu$ as $h \to \infty$ and, by the characterization of the local weak* convergence (see Theorem 1.113),

(4.92) $$\mu(K) \geqslant \limsup_{h\to\infty} \mu_h(K) \text{ for each compact set } K\,,$$

(4.93) $$\mu(A) \leqslant \liminf_{h\to\infty} \mu_h(A) \text{ for each open set } A\,.$$

We are going to show that

(4.94) $$\mu(F) = \lambda(F) \text{ for each Borel set } F \subset \mathbb{R}^2\,.$$

Let us begin to show that μ is concentrated on E, that is

(4.95) $$\mu(\mathbb{R}^2 \setminus E) = 0 \text{ and } \operatorname{spt}(\mu) = E\,.$$

Observe that, by definition, since $P_h \subset C_k \times C_k$ if $h \geqslant k \geqslant 1$, then

(4.96) $$P_h \cap F = P_h \cap (\cap_{k=1}^m C_k \times C_k) \cap F \quad \forall\, h \geqslant m \geqslant 1,\ F \subset \mathbb{R}^2\,,$$

and, for each $h \geqslant k \geqslant 1$, $s = 1,\dots,4^k$

(4.97) $$4 = \mathcal{H}^1(P_h) = \mathcal{H}^1(P_h \cap (C_k \times C_k)) = 4^k\,\mathcal{H}^1(P_h \cap Q_{k,s})$$

By (4.92), (4.93) and (4.96), it follows that, for each compact set $K \subset \mathbb{R}^2$ and integer m,

$$\begin{aligned}\mu(\overset{\circ}{K}) &\leqslant \liminf_{h\to\infty} \mathcal{H}^1 \llcorner P_h(\overset{\circ}{K}) \leqslant, \limsup_{h\to\infty} \mathcal{H}^1 \llcorner P_h(K)\\ &= \limsup_{h\to\infty} \mathcal{H}^1 \llcorner P_h\left(K \cap (\cap_{k=1}^m C_k \times C_k)\right)\\ &\leqslant \mu\left(K \cap (\cap_{k=1}^m C_k \times C_k)\right).\end{aligned}$$

Passing to the limit, when $m \to \infty$, in the previous inequality, we get that

$$\mu(\overset{\circ}{K}) \leqslant \mu\left(K \cap E\right) \text{ for each compact set } K \subset \mathbb{R}^2,$$

and, since E is compact, this implies that

$$\mu(\mathbb{R}^2) \leqslant \mu(E) < \infty$$

and then (4.95) follows. Let us now prove that

$$\mu\left(B((x,y),r_k)\right) = \mu\left(E \cap B((x,y),r_k)\right) = \frac{1}{4^{k-1}} \text{ for each } k \geqslant 1,\ (x,y) \in E. \tag{4.98}$$

The first identity immediately follows by (4.95). By (4.89), (4.90) and (4.95), we can infer that, for each $(x,y) \in E$, for each k there exists a unique $1 \leqslant \bar{s} \leqslant 4^k$ such that

$$\mu\left(E \cap B((x,y),r_k)\right) = \mu\left(E \cap Q_{k,\bar{s}}\right) = \mu\left(Q_{k,\bar{s}}\right). \tag{4.99}$$

By (4.92) and (4.97), it follows that

$$\mu\left(Q_{k,\bar{s}}\right) \geqslant \limsup_{h\to\infty} \mathcal{H}^1\left(P_h \cap Q_{k,\bar{s}}\right) = 4^{1-k}.$$

The reverse inequality can be obtained noticing that, by (4.82), we can fatten the closed cube $Q_{k,\bar{s}}$ by an open cube $\tilde{Q}_{k,\bar{s}}$ in such a way, for each $s = 1, \dots, 4^k$,

$$Q_{k,\bar{s}} \subset \tilde{Q}_{k,\bar{s}} \text{ and } \tilde{Q}_{k,\bar{s}} \cap Q_{k,s} \neq \emptyset \text{ if and only if } s = \bar{s}.$$

Thus, by (4.95), (4.93) and (4.97),

$$\begin{aligned}\mu\left(Q_{k,\bar{s}}\right) &= \mu\left(E \cap Q_{k,\bar{s}}\right) = \mu\left(E \cap \tilde{Q}_{k,\bar{s}}\right)\\ &= \mu\left(\tilde{Q}_{k,\bar{s}}\right) \leqslant \liminf_{h\to\infty} \mu_h\left(\tilde{Q}_{k,\bar{s}}\right)\\ &= \liminf_{h\to\infty} \mu_h\left(Q_{k,\bar{s}}\right) = 4^{1-k}.\end{aligned}$$

By the two previous inequalities and (4.99), the second identity in (4.96) also follows. By (4.98), (4.88) and (4.90), we can infer that

$$\begin{aligned}&\mu\left(B((x,y),r_k)\right)\\ &\quad = \lambda\left(B((x,y),r_k)\right) = 4^{1-k} \quad \forall\,(x,y) \in \operatorname{spt}(\mu) = E,\ k = 1,2,\dots.\end{aligned} \tag{4.100}$$

Let us now prove that

$$\mu << \lambda \tag{4.101}$$

or, that is equivalent by the differentiation for positive measures (see Theorem 2.15),

$$\underline{D}_\lambda \mu(x,y) := \liminf_{r\to 0^+} \frac{\mu\left(B((x,y),r)\right)}{\lambda\left(B((x,y),r)\right)} < \infty \quad \mu\text{-a.e. } (x,y) \in \mathbb{R}^2. \tag{4.102}$$

By (4.100), we immediately get that

$$\liminf_{r\to 0^+} \frac{\mu\,(B((x,y),r))}{\lambda\,(B((x,y),r))} \leqslant \liminf_{k\to\infty} \frac{\mu\,(B((x,y),r_k))}{\lambda\,(B((x,y),r_k))} = 1 \quad \forall\,(x,y)\in E\,,$$

which implies, by (4.95), (4.102) and then (4.101). Since it is trivial that the sequence of sets

$$E_k(x,y) := B((x,y),r_k) \quad k=1,2,\dots,\ (x,y)\in\mathbb{R}^2$$

is a differentiation basis for λ, by applying Theorem 2.23 and (4.100)

$$\exists\, 1 = \lim_{k\to\infty} \frac{\mu(E_k(x,y))}{\lambda(E_k(x,y))} = D_\lambda\mu(x,y) \quad \lambda\text{-a.e. } (x,y)\in \mathrm{spt}(\lambda)\,,$$

which implies (4.94).

4.4. **Extensions to metric spaces.**

The notion of rectifiable set in a metric space was already introduced by Federer [Fe, 3.2.14].

Definition 4.29. A set $\Gamma\subset(X,d)$ is said to be *countably $\mathcal{H}^k$-rectifiable* if

$$\Gamma = \Gamma_0 \cup \left(\cup_{i=1}^\infty f_i(A_i)\right)\,,$$

where $\mathcal{H}^k(\Gamma_0)=0$, $A_i\subset\mathbb{R}^k$ are $\mathcal{L}^k$-measurable and $f_i:\ A_i\subset(\mathbb{R}^k,\|\cdot\|_{\mathbb{R}^k})\to(X,d)$ $(i=1,2,\dots,)$ are Lipschitz functions.

Definition 4.30. A metric space (X,d) is said to be *purely $\mathcal{H}^k$- unrectifiable* if for each Lipschitz function $f:\ A\subset(\mathbb{R}^k,\|\cdot\|_{\mathbb{R}^k})\to(X,d)$,

$$\mathcal{H}^k(f(A)) = 0\,.$$

A quite general structure rectifiability result can be obatined for the $\mathcal{H}^1$-rectifiability (see, for instance, [AT, Theorem 4.4.8]).

Theorem 4.31. *If (X,d) is complete, $\Gamma\subset X$ is closed and connected, and $\mathcal{H}^1(\Gamma)<\infty$, then there exist countably many Lipschitz curves $\gamma_h:[0,1]\to\Gamma$ $(h=1,2,\dots)$ such that*

$$\mathcal{H}^1\left(\Gamma\setminus\cup_{h=1}^\infty\gamma_h([0,1])\right) = 0\,.$$

In particular Γ is countably $\mathcal{H}^1$-rectifiable.

The study of higher $\mathcal{H}^k$-rectifiability with $k\geqslant 2$ in a metric space is much harder. A systematic study of rectifiable sets in general metric spaces was made by Ambrosio and Kirchheim [AK] in 2000. However, the definitions they used are not always appropriate in some remarkable class of metric spaces such as the one called *Carnot groups* or also *sub-Riemannian stratified groups.* Indeed Ambrosio and Kirchheim proved the following result.

Theorem 4.32. *(*[AK, Theorem 7.2]*) The first Heisenberg group $(\mathbb{H}^1,d)$ is purely k-unrectifiable for $k=2,3,4$, for each invariant distance d.*

Therefore, taking the previous unrectifiability results into account, a new suitable notion of rectifiability in Carnot groups is needed, better fitting the new geometry. This study is still object of the current research and an account can be found in [SC2].

5. An introduction to minimal surfaces and sets of finite perimeter. ([AFP, G, G2, Mag, MM])

Motivation: An introduction to the so-called Plateau's problem for non-parametric minimal surfaces as far as the problem of existence, uniqueness and regularity is concerned. An introduction to the sets of finite perimeter and their relationships with the minimal surfaces.

5.1. **Plateau problem: nonparametric minimal surfaces in $\mathbb{R}^n$, area functional and its minimizers.** Here we are going to deal with the problem of least area for the so-called *non parametric hypersurfaces* in $\mathbb{R}^n$, that is hypersurfaces which are graphs of functions. More precisely we will consider an hypersurface $S \subset \mathbb{R}^n$ with

$$S = S_u := \{(z, u(z)) : z \in \bar{\omega}\}$$

where $u \in \mathbf{C}^1(\bar{\omega})$ and $\omega \subset \mathbb{R}^{n-1}$ is a bounded open set with smooth boundary. Non parametric minimal surfaces are a particular case of the general theory of (parametric) minimal surfaces, where also surfaces satisfying Definition 4.1 are allowd and they may be not graphs. A recent account of the development and open problems of this theory can be found in [Pe].

By (3.74), the $(n-1)$-dimensional area of graph S_u is given by

$$\mathscr{A}(u) := \mathcal{H}^{n-1}(S_u) = \int_\omega \sqrt{1 + |\nabla u|^2}\, dx \tag{5.1}$$

and this formula holds true for each funnction $u \in Lip(\omega)$. Thus we can define the classical *area functional*

$$\mathscr{A} \equiv \mathscr{A}(\cdot, \omega) : Lip(\omega) \to [0, \infty), \quad \mathscr{A}(u) = \mathscr{A}(u, \omega) := \int_\omega \sqrt{1 + |\nabla u|^2}\, dx\,.$$

This gives rise to the classical *Plateau's problem*, that is to show the existence of an area minimizing hypersurface with a given boundary. More precisely, if we fix a boundary datum $g : \partial\omega \to \mathbb{R}$ we are concerned with the geometric variational problem of the calculus of variations, also called *Dirichlet problem*,

$$\min \{\mathscr{A}(u, \omega) : u \in Lip(\omega, g)\}\ , \tag{PP}$$

where $Lip(\omega, g)$ denotes the class of competitors functions

$$Lip(\omega, g) := \{u \in Lip(\bar{\omega}) : u = g \text{ on } \partial\omega\}\ ,$$

which is supposed to be nonempty. As usual in the calculus of variations, the main two questions about (PP) concerns:

(EUPP) existence and uniqueness of a *minimizer* u_0 of (PP), that is whether there is a function $u_0 \in Lip(\omega, g)$ satisfying

$$\mathscr{A}(u_0) \leqslant \mathscr{A}(u) \quad \forall\, u \in Lip(\omega, g)$$

and whether u_0 is unique;

(RPP) regularity of minimizers, that is whether a minimizer is (locally) regular in ω, or also (globally) regular in $\bar{\omega}$ provided that both datum g and boundary $\partial\omega$ are regular.

Here we will mainly deal with the existence of minimizers, and we will only mention later some regularity results as far as minimal boundaries, which applies to problem (PP), too.

Existence problem for (PP) essentially was studied by means of two strategies (see [G2, Introduction] for a more complete and interesting account of this issue).

- The first strategy is by studying the associated *Euler-Lagrange equation* to area functional $\mathscr{A}$. Namely

  Exercise: prove that, given $\varphi \in \mathbf{C}^\infty(\omega)$ and u_0 minimizer of (PP), let $\mathbb{R} \ni t \mapsto a(t) := \mathscr{A}(u_0 + t\varphi)$, then $t = 0$ is a minimum point of a and

$$\exists\, 0 = a'(0) = \int_\omega \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}}\, dz\,.$$

  The previous identity yields the associated Euler-Lagrange equation to area functional $\mathscr{A}$, called *minimal surface equation* and, in divergence form, it reads as

(MSE)
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad \text{in } \omega\,.$$

  (MSE) is a nonlinear PDE equation, which has been deeply studied for two centuries and half long with regard to existence, uniqueness and regularity of its solutions. Then one transfers those results about solutions of (MSE) to minimizers of area functional (see [G, G2, MM]).
- The second strategy consists in showing directly the existence of minimizers for the area functional without studying its associated Euler-Lagrange, that is the so-called *direct methods of the calculus of variations*. In the following we will follow this strategy first for problem (PP) and then for the generalized problem (PP) dealing with sets of finite perimeter.

Historical notes: Plateau's problem originates much before with Lagrange who in 1762 studied it, derived the minimal surface equation (MSE) and gives a foundation of the calculus of variations. We have also to acknoweledge that Euler in 1741 already exibited an example of minimal surface by means of the *catenoid* (see Example 5.6). The historical development of the theory of minimal surfaces involves many of the greatest mathematicians of their time and an account can be found in [Pe].

5.2. **Direct method of the calculus of variations and application to the existence of minimizers for the Plateau problem.** One of the general results applying the direct methods in calculus of variations is the following generalized Weierstrass theorem for the existence of minimizers.

Theorem 5.1 (Generalized Weierstrass theorem)**.** *Let (X, τ) be a topological space and let $F : X \to (-\infty, +\infty]$ be a functional such that*

(i) *F is sequentially lower semicontinuous (slsc), i.e. for each $x \in X$ and sequence $(x_h)_h \subset X$ with* $\lim_{h\to\infty} x_h = x$, *then*

$$F(x) \leqslant \liminf_{h\to\infty} F(x_h)\,;$$

(i) *F is sequentially coercive, that is for each $t \in \mathbb{R}$ there exists a sequentially compact set $K_t \subset X$ such that*

$$\{x \in X : F(x) \leqslant t\} \subseteq K_t \,.$$

Then $\exists \min_X F$.

Proof. If $F \equiv +\infty$, the conclusion is trivial, otherwise let $(x_h)_h \subset X$ be such that

$$\lim_{h\to\infty} F(x_h) = m := \inf_X F < +\infty \,.$$

If $t > m$, there exist $\bar{h}$ and a sequentially compact set K_t such that

$$x_h \in \{x \in X : F(x) \leqslant t\} \subseteq K_t \quad \forall\, h > \bar{h} \,.$$

Up to a subsequence, we can suppose, by claim (ii), that there exist $x \in K_t$ such that $\lim_{h\to\infty} x_h = x$. Let us show

$$F(x) = m \,,$$

which will imply our conclusion. Indeed, by claim (i),

$$F(x) \leqslant \liminf_{h\to\infty} F(x_h) = \lim_{h\to\infty} F(x_h) = m = \inf_X F \,,$$

and we reach the desired conclusion. □

A classical application of the direct methods of the calculus of variations to problem (PP) can be given by the so-called *bounded slope condition* property for boundary datum g, which goes back to Hilbert, and we recall here.

Definition 5.2. We say that a function $g : \partial\omega \to \mathbb{R}$ satisfies the *bounded slope condition with constant* $Q > 0$ (Q-B.S.C. for short, or simply B.S.C. when the constant Q does not play any role) if for every $z_0 \in \partial\omega$ there exist two affine functions $w^+_{z_0}$ and $w^-_{z_0}$ such that

$$\begin{aligned} &w^-_{z_0}(z) \leqslant g(z) \leqslant w^+_{z_0}(z) \quad \forall z \in \partial\omega, \\ &w^-_{z_0}(z_0) = g(z_0) = w^+_{z_0}(z_0) \\ &Lip(w^-_{z_0}) \leqslant Q \quad \text{and} \quad Lip(w^+_{z_0}) \leqslant Q, \end{aligned}$$

where $Lip(w)$ denotes the Lipschitz constant of w.

We also recall that a set $\omega \subset \mathbb{R}^{n-1}$ is said to be *uniformly convex* if there exist a positive constant $C = C(\omega)$ and, for each $z_0 \in \partial\omega$, a hyperplane Π_{z_0} passing through z_0 such that

$$|z - z_0|^2 \leqslant C\, d(z, \Pi_{z_0}) \quad \forall z \in \partial\omega,$$

where $d(z, \Pi_{z_0}) := \inf\{|z - w| \mid w \in \Pi_{z_0}\}$.

Remark 5.3. We collect here some facts on the B.S.C.

a) If $g : \partial\omega \to \mathbb{R}$ satisfies the B.S.C. and is not affine, then ω has to be convex (see [G2, page 20]) and g is Lipschitz continuous on $\partial\omega$. Moreover, if $\partial\omega$ has flat faces, then g has to be affine on them.

This property seems to say that the B.S.C. is a quite restrictive assumption. Anyhow the following one, due to M. Miranda [Mi] (see also [G2, Theorem 1.1]), shows that the class of functions satisfying the B.S.C. on a uniformly convex set is quite large.

b) Let $\omega \subset \mathbb{R}^n$ be open, bounded and uniformly convex; then every $g \in C^{1,1}(\mathbb{R}^n)$ satisfies the B.S.C. on $\partial\omega$.

Theorem 5.4. *Let ω be a bounded open set in $\mathbb{R}^{n-1}$ and assume that $g : \partial\omega \to \mathbb{R}$ satisfies B.S.C. with constant $Q > 0$. Then problem* (PP) *has a unique minimizer $u_0 \in Lip(\omega, g)$. Moreover u_0 satisfies the estimate*

$$Lip(u_0) \leqslant Q\,.$$

Proof. See, for instance, [G2, Theorem 1.2]. □

Remark 5.5. In Theorem 5.4 we obtained existence and uniqueness of minimizers under special assumptions of convexity on the domain ω. If we want to weaken these conditions, so as to treat more general domains ω, we could use new comparison functions, more general than the affine functions of the B.S.C. Indeed it is possible to prove, by means of the *barriers method*, the existence and uniqueness of a minimizer for (PP) in $Lip(\omega, g)$, provided that bounded open set ω has a boundary of class $\mathbf{C}^2$, with *non-negative mean curvature* and g is of class $\mathbf{C}^2$ (see [G2, Theorem 1.6]). Let us recall that if ω is an open bounded set of class $\mathbf{C}^2$ and convex, then its boundary has non-negative mean curvature (see [G2, Page 29]). The condition on non-negative mean curvature on the boundary is almost necessary. In fact, one can prove that if the mean curvature is negative at some point of boundary $\partial\omega$, then there exists a regular function g for which the area functional has no minimum in $Lip(\omega, g)$ (see [G2, Theorem 1.7]). We will show by means of an example such an eventuality.

Example 5.6 (Non-existence for Plateau's problem)**.** Let $n = 3$ and let $\omega \subset \mathbb{R}^2$ be the annulus

$$\omega := \{z = (x, y) \in \mathbb{R}^2 :\ \varrho < |z| < R\} \tag{5.2}$$

with $0 < \varrho < R$ given. Consider problem (PP) with boundary datum

$$g(z) = \begin{cases} 0 & \text{if } |z| = R \\ M & \text{if } |z| = \varrho\,, \end{cases} \tag{5.3}$$

with $M > 0$. We will show that this problem admits no minimizer when M is large enough.

We begin by proving that, if a minimizer exists, then there exists a rotationally invariant one. To this aim, it is enough to prove that for any $u \in Lip(\omega)$ we have

$$\int_\omega \sqrt{1 + |\nabla \tilde{u}|^2}\, d\mathcal{L}^2 \leqslant \int_\omega \sqrt{1 + |\nabla u|^2}\, d\mathcal{L}^2 \tag{5.4}$$

where, after setting R_θ to be the rotation in $\mathbb{R}^2$ of an angle θ, that is the linear isometry $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$$R_\theta(x, y) := (\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y)\,,$$

we define the rotationally symmetric function $\tilde{u} : \omega \to \mathbb{R}$ by

$$\tilde{u}(z) := \fint_0^{2\pi} (u \circ R_\theta)(z)\, d\theta = \fint_0^{2\pi} u(|z|\cos\theta, |z|\sin\theta)\, d\theta\,. \tag{5.5}$$

Indeed, when $u \in Lip(\omega)$ one has

Exercise:

$$\nabla(u \circ R_\theta) = R_{-\theta} \circ (\nabla u) \circ R_\theta \qquad \mathcal{L}^2\text{-a.e. in } \omega\,. \tag{5.6}$$

By (5.6) and since R_θ and $R_{-\theta}$ are isometries, it follows that, for $\mathcal{L}^2$-a.e. $z \in \omega$,

$$\begin{aligned} |\nabla\tilde{u}(z)| &= \left| ⨍_0^{2\pi} \nabla(u \circ R_\theta)(z)\, d\theta \right| \\ &= \left| ⨍_0^{2\pi} \left(R_{-\theta} \circ \nabla u \circ R_\theta\right)(z)\, d\theta \right| \\ &\leqslant ⨍_0^{2\pi} \left|R_{-\theta} \circ \nabla u \circ R_\theta\right|(z)\, d\theta \\ &= ⨍_0^{2\pi} \left|\nabla u\right|(z)\, d\theta = |\nabla u(z)|\,. \end{aligned} \tag{5.7}$$

Thus, by (5.7), (5.4) is proved . Moreover, it is not difficult to show that

$$\tilde{u}_{|\partial\omega} = g \tag{5.8}$$

for any $u \in Lip(\omega, g)$, that is that also $\tilde{u} \in Lip(\omega, g)$.

We are now going to exclude the existence of rotationally invariant minimizers.

Let $u(z) = u(x,y) = v\left(\sqrt{x^2+y^2}\right)$ where

$$v \in Lip^*(\varrho, R) := \{v \in Lip([\varrho, R]) : \, v(\varrho) = M \text{ and } v(R) = 0\}\,.$$

Then it is easy to set that $u \in Lip(\omega, g)$ and

$$|\nabla u(z)| = |v'(|z|)| \quad \mathcal{L}^2\text{-a.e. } z \in \omega\,.$$

Thus, we obtain

$$\int_\omega \sqrt{1+|\nabla u|^2}\, d\mathcal{L}^2 = 2\pi \int_\varrho^R r\sqrt{1+v'(r)^2}\, dr \;=: L(v)\,. \tag{5.9}$$

We are going to show that for $M \gg 1$ the functional L does not admit minimizers in the one-dimensional class $Lip^*(\varrho, R)$.

Suppose now that v is a minimizer in $Lip^*(\varrho, R)$.

Exercise: Prove that function

$$[\varrho, R] \ni r \mapsto \frac{r\, v'(r)}{1+v'(r)^2} \text{ is absolutely continuous}$$

and

$$\frac{d}{dr}\left(\frac{r\, v'(r)}{1+v'(r)^2}\right) = 0 \quad \mathcal{L}^1\text{-a.e. } r \in [\varrho, R]\,. \tag{5.10}$$

(*Hint:* Let us consider the Euler-Lagrange equation associated to L: let $\varphi \in \mathbf{C}_c^1((\varrho, R))$ be given and let $l(t) := L(v + t\,\varphi)$ if $t \in \mathbb{R}$; then $l'(0) = 0$ and deduce the desired conclusion.)

By (5.10), it follows that

$$\frac{r\, v'(r)}{\sqrt{1+v'(r)^2}} = c \quad \forall\, r \in [\varrho, R]\,, \tag{5.11}$$

for a suitable $c \in \mathbb{R}$. In particular for $r \in [\varrho, R]$

$$|c| = \left| \frac{r\, v'(r)}{\sqrt{1+v'(r)^2}} \right| \leqslant |r|$$

and so $|c| \leqslant \varrho$. From (5.11) and taking into account sgn $v' =$ sgn c we obtain

$$v'(r) = \frac{c}{\sqrt{r^2-c^2}} = \frac{\operatorname{sgn} c}{\sqrt{(r/c)^2-1}} \quad \mathcal{L}^1\text{-a.e. } r \in [\varrho, R]\,. \tag{5.12}$$

By (5.12), v has to be monotone, and since $v(\varrho) = M > v(R) = 0$, v is not increasing. Thus $C < 0$, and, without loss of generality, by replacing c with $-c$,the solution is given by

$$\begin{aligned} v(r) &= \int_r^R \frac{d\sigma}{\sqrt{(\sigma/c)^2-1}} = \int_{r/c}^{R/c} \frac{c\, d\sigma}{\sqrt{\sigma^2-1}} \\ &= c\,(\operatorname{arccosh}(R/c) - \operatorname{arccosh}(r/c)) \text{ if } \varrho \leqslant r \leqslant R\,. \end{aligned} \tag{5.13}$$

where, by definition, arccosh : $[1, \infty) \to [0, \infty)$ is the inverse function of hyperbolic cosine function

$$\cosh :\ [0,\infty) \to [1,\infty),\ \cosh(t) := \frac{e^t + e^{-t}}{2}$$

and it can be explicitely represented as

$$\operatorname{arccosh}(s) = \log\left(s + \sqrt{s^2-1}\right) \text{ if } s \geqslant 1\,.$$

Moreover the constant $0 < c \leqslant \varrho$ has to be chosen in order that $v(\varrho) = M$. Observe that we can equivalently represent v as

$$v(r) = c\, \log \frac{R+\sqrt{R^2-c^2}}{r+\sqrt{r^2-c^2}} \text{ if } \varrho \leqslant r \leqslant R\,, \tag{5.14}$$

and

$$M = v(\varrho) = c\, \log \frac{R+\sqrt{R^2-c^2}}{\varrho+\sqrt{\varrho^2-c^2}} \leqslant \sup_{c\in(0,\varrho]} \left(c\, \log \frac{R+\sqrt{R^2-c^2}}{\varrho+\sqrt{\varrho^2-c^2}} \right)\,. \tag{5.15}$$

Exercise: Prove that

$$\sup_{c\in(0,\varrho]} \left(c\, \log \frac{R+\sqrt{R^2-c^2}}{\varrho+\sqrt{\varrho^2-c^2}} \right) = \varrho\, \log \frac{R+\sqrt{R^2-\varrho^2}}{\varrho} =: M_0(\varrho, R)\,. \tag{5.16}$$

(*Hint:* Prove that the function $(0, \varrho] \ni c \mapsto c\, \log \dfrac{R+\sqrt{R^2-c^2}}{\varrho+\sqrt{\varrho^2-c^2}}$ is nondecreasing.)

By (5.14), (5.15) and (5.16), it follows that **problem** (PP) **can be solved only if** $M < M_0$, when ω is the open set in (5.2) and g is the boundary datum in (5.3) . In the limit case $M = M_0$ we have $c = \varrho$ and,

$$|\nabla u(x,y)| = \left| v'\left(\sqrt{x^2+y^2}\right) \right| = \frac{\varrho}{\sqrt{x^2+y^2-\varrho^2}}$$

becomes infinite on the internal circumference and then u **is not admissible because** $u \notin Lip(\omega)$. However a minimizer exists in a larger class of competitors, namely in the *Sobolev space* $W^{1,1}(\omega)$, as proved in the following exercise.

Exercise: Let us define the space of functions

$$AC^*(\varrho, R) := \{v \in AC([\varrho, R]) : v(\varrho) = M_0 \text{ and } v(R) = 0\} .$$

(i) Prove that functional L in (5.9) is well-defined on $AC^*(\varrho, R)$, that is, $L : AC^*(\varrho, R) \to [0, \infty)$.

(ii) Let v_0 be the function in (5.14) with $c = \varrho$. Prove that v_0 is a minimizer of functional $L : AC^*(\varrho, R) \to [0, \infty)$, that is, $v_0 \in AC^*(\varrho, R)$ and

$$L(v_0) \leqslant L(v) \quad \forall\, v \in AC^*(\varrho, R) .$$

(iii) Let

$$u_h(z) := v_h(|z|),\ u_0(z) := v_0(|z|) \text{ if } z \in \omega . \tag{5.17}$$

where

$$v_h(r) := c_h\, \varrho \log \frac{R + \sqrt{R^2 - \varrho^2 + 1/h}}{r + \sqrt{r^2 - \varrho^2 + 1/h}} \text{ if } \varrho \leqslant r \leqslant R$$

and the constant c_h is chosen in such a way that

$$v_h(\varrho) = M_0 .$$

Prove that

$$(u_h)_h \subset \mathbf{C}^1(\bar{\omega}) \cap Lip(\omega, g), \tag{5.18}$$

$$u_h \to u_0 \text{ uniformly on } \bar{\omega},\ \nabla u_h \to w \text{ in } (L^1(\omega))^2 \tag{5.19}$$

where

$$w(z) := v_0'(|z|) \frac{z}{|z|} \text{ if } \varrho < |z| < R .$$

Moreover

$$L(v_0) = \lim_{h\to\infty} L(v_h) = \lim_{h\to\infty} \mathscr{A}(u_h, \omega) = \inf\{\mathscr{A}(u, \omega) : u \in Lip(\omega, g)\} , \tag{5.20}$$

where g is the boundary datum in (5.3) with $M = M_0$.

(iv) Prove that

$$u_0 \in W^{1,1}(\omega) \cap \mathbf{C}^0(\bar{\omega}) ,$$

that is, by definition, $u_0 \in \mathbf{C}^0(\bar{\omega})$, $u_0 \in L^1(\omega)$ and there exists its *weak gradient*

$$Du_0(z) = w(z) \text{ if } \varrho < |z| < R ,$$

with $Du_0 \in (L^1(\omega))^2$. Moreover $u_0 \notin \mathbf{C}^1(\bar{\omega})$ as well $u_0 \notin Lip(\omega)$, even if

$$L(v_0) = \int_\omega \sqrt{1 + |Du_0|^2}\, dx .$$

If $M > M_0$ **there is no solution to problem** (PP). In this case, one can prove that, by means of the direct methods and by replacing the space of Lipschitz continuous functions with the one of *bounded variation functions* , the minimal surface is given by the graph of the solution u corresponding to the limit value M_0, plus the portion of the vertical cylinder having for base the internal circumference of radius ϱ, that lies between the levels M_0 and M (see [G, Chapter 14]).

Finally let us notice that, in the case $M < M_0$, the function v in (5.13) is the inverse function of

$$r : [0, M] \to [\varrho, R], \ r(v) := c \cosh\left(\frac{v-b}{c}\right)$$

where $b := c \operatorname{arccosh}(R/c)$, which is an arc of *catenary* joining the points $(0, R)$ and (M, ϱ) in the plane v, r. By rotation about the v-axis, it generates a surface called *catenoid*, graph of u, which is a surface of revolution of minimal area. Surfaces of revolution of minimal area was an issue very studied in the history of calculus of variations, starting from Euler. An account of this fascinating development can be found in [GH, Chap. 5, Sect. 2.4, Example 5].

Historical notes: [MM] Variational problems concerning manifolds, one or more dimensional, immersed in an Euclidean space are among the most classical ones. We mean that they have been considered since Bernoulli's time and have not obtained a general satisfactory treatment until the the 1950s. At the start of the last century very interesting new ideas about variational problems for surfaces, are contained in the thesis of Lebesgue "Integrale, Longueur, Aire". Of the same period of time, are the interesting papers of Tonelli about the length of the curves. In the 1930s appeared the relevant series of papers by Douglas and Radò about the Plateau Problem, together with some interesting contributions of Tonelli concerning variational problems with two independent variables. The school of Tonelli, particularly Cesari, worked at the problem of a definition of the surface area, good from the variational point of view. But it was only in the 1950s that new definitions of surfaces were introduced and used for a general treatment of classical variational problems like the isoperimetric property of the sphere and the Plateau Problem. In the new approaches, like Reifenberg surfaces, Federer-Fleming integral currents, De Giorgi perimeters and Almgren varifolds, ideas from the Modern Algebra, General Measure Theory and Distribution Theory are used together with the classical arguments from Differential Geometry and Real Variable Functions Theory.

5.3. **Sets of finite perimeter, space of bounded variation functions and their main properties; sets of minimal boundary.** We are going to introduce an alternative approach to the theory minimal surfaces, which applies to *hypersurfaces* of $\mathbb{R}^n$, that is surfaces of topological dimension $n-1$ and which extend the one studied before for non-parametric minimal surfaces. The main idea is that a hypersurface in $\mathbb{R}^n$ is meant as boundary of a set $E \subset \mathbb{R}^n$ whose characteristic function χ_E has *bounded variation*, namely E is a set of *finite perimeter*. The notion goes back to De Giorgi, who introduced it in the pioneering papers [DG1, DG2], strongly inspired by some previous ideas of Caccioppoli (see [A2] for an interesting account of Euclidean sets of finite perimeter). Caccioppoli's primitive idea, then refined by De Giorgi through sets of finite perimeter, considered *oriented hypersurfaces*, which (at least locally) are boundaries of sets, and exploited techniques of measure theory. Let us begin to stress some benefits for the introduction of sets of finite perimeter by means of two least-area problems.

Problem 1 (Plateau's problem for general domains). We can give a formulation of Plateau' problem (PP) in terms of boundary of a set in $\mathbb{R}^{n+1}$ as follows. Let $\omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary and let denote with Ω and $\bar{\Omega}$ respectively the open and closed cylinders

$$\Omega := \omega \times \mathbb{R}, \quad \bar{\Omega} := \bar{\omega} \times \mathbb{R}\,. \tag{5.21}$$

If $v : \bar{\omega} \to \mathbb{R}$, let $E_v \subset \mathbb{R}^{n+1}$ the (closed) *subgraph induced by* v, that is

$$E_v := \left\{(z, x_{n+1}) \in \bar{\Omega} : \, x_{n+1} \leqslant \, v(z)\right\} \tag{5.22}$$

and let $S_v \subset \mathbb{R}^{n+1}$ be the *graph of* v, that is

$$S_v := \left\{(z, x_{n+1}) \in \bar{\Omega} : \, x_{n+1} = v(z)\right\} \,. \tag{5.23}$$

Let u, $g : \bar{\omega} \to \mathbb{R}$ be Lipschitz continuous functions. Then it is easy to see that

$$\partial E_u \cap \Omega = \, S_u \cap \Omega$$

and we can mean the boundary condition $u = \, g$ on $\partial\omega$ as

$$E_u = \, E_g \text{ in } \mathbb{R}^{n+1} \setminus \Omega\,. \tag{5.24}$$

Therefore Plateau's problem (PP) can be formulated and extended in terms of sets as follows

$$\min\left\{\mathcal{H}^n(\partial E_u \cap \Omega) : \, u \in X, E_u \text{ satisfies (5.24) }\right\}$$

where X is a suitable set of functions to be chosen. The benefit of this formulation is the chance to get existence for more general regular domains ω than the ones strict convex allowed in Theorem 5.4 and Remark 5.4 (see also Example 5.6). Indeed this is the case since, by the *relaxation method*, one can find out that suitable class X of competitors is the the space of functions of bounded variations on ω (or, equivalently, sets E_u of finite perimeter in cylinder Ω) and then, by a direct methods of the calculus of variation, to get the existence (see [G, Sect. 14.4]).

Problem 2 (Isoperimetric problem). The classical *isoperimetric problem* is the most celebrated problem of least-area and, likely, one of the earliest problem in this argument: it asks to find out a plane figure of the largest possible area whose boundary has a specified length. It can be extended to higher dimensions as follows: if $E \subset \mathbb{R}^n$ and

$$|E| := \mathcal{L}^n(E), \quad \mathcal{H}^{n-1}(\partial E)\,,$$

respectively denote the n-dimensional volume of E and the $(n-1)$-dimensional area (we will call also perimeter) of ∂E, to find out a set E_{iso} with maximum n-volume, among sets $E \in X$ with $|E| < \, \infty$ and fixed perimeter $\mathcal{H}^{n-1}(\partial E) = c$, where X is a suitable class of admissible sets for the n-volume and perimeter functions, stable by translations an dilations, to be chosen (for instance, X could be the class of sets with $\mathbf{C}^1$ regular boundary).

It is also well-known that the isoperimetric problem is equivalent to the so-called *isovolumetric problem*, that is to find out a set E_{iso} with minimum perimeter, among sets $E \in X$ with fixed volume $|E| = \, c$. Indeed, by means of the *isoperimetric inequality*, that is the inequality

$$\min\left\{|E|^{(n-1)/n}, \, |\mathbb{R}^n \setminus E|^{(n-1)/n}\right\} \leqslant \, C\,\mathcal{H}^{n-1}(\partial E) \tag{II}$$

which holds for a suitable positive constant $C > 0$ and for each set $E \in X$, the isoperimetric problem is equivalent to the isovolumetric problem

$$\text{(IP)} \qquad m = \min\left\{\mathcal{H}^{n-1}(\partial E) : E \in X, |E| = 1\right\}$$

since the set functions

$$X \ni E \mapsto |E|^{(n-1)/n} \text{ and } X \ni E \mapsto \mathcal{H}^{n-1}(\partial E)$$

are homogeneous of degree $n-1$. Moreover, if problem (IP) has a solution, the value $C_{\text{iso}} = 1/m$ is the minimum constant C for which (II) turns to be true.

It is also well-known that balls are the only solutions of problem (IP) when X is the class of sets with C^1 boundary. Thus we can explicitely calculate the value of C_{iso} and it turns out to be

$$\text{(5.25)} \qquad C_{\text{iso}} = \left(n^n\,\alpha_n^*\right)^{1/(1-n)} ,$$

where $\alpha_n^* := |B(0,1)|$. Observe that the isoperimetric constant (5.25) could depend on the class X. However De Giorgi [DG3] proved the minimality of balls in the largest admissible class X, namely the sets of finite perimeter.

An other important feature that class X of competitors in problems 1 and 2 has to satisfy is the sequential compactness with respect to a suitable topology τ, in order to apply the direct methods of the calculus of variations. It is quite clear that neither the class of sets with boundary of class $\mathbf{C}^1$ nor the one of class Lipschitz are appropriate for this goal. For instance, let us assume that $(E_h)_h$ is the following sequence

$$E_h = E_{u_h} ,$$

where $\omega \subset \mathbb{R}^2$ is the domain in (5.2) and $(u_h)_h$ is the sequence of functions defined in (5.17). Then, by the previous exercise, it turns out that $(u_h)_h$ is a minimizing sequence for Plateau's problem (PP) in the class $Lip(\omega, g)$ and, if $\Omega := \omega \times \mathbb{R}$,

$$\lim_{h\to\infty} \mathcal{H}^2\left(\partial E_h \cap \Omega\right) = \lim_{h\to\infty} \int_\omega \sqrt{1 + |\nabla u_h|^2}\, dx = \int_\omega \sqrt{1 + |Du_0|^2}\, dx$$

but u_0 is not neither in $\mathbf{C}^1(\bar\omega)$ nor in $Lip(\omega)$, even though $(u_h)_h$ uniformly converges to u_0.

The fundamental result to take into account for the introduction of sets of finite perimeter is the Gauss-Green theorem, which holds for sets with regular boundary.

Theorem 5.7 (Gauss-Green). *Assume that $E \subset \mathbb{R}^n$ is an open set with its boundary ∂E of class $\mathbf{C}^1$ and let $g = (g_1, \dots, g_n) \in \mathbf{C}^1_c(\mathbb{R}^n, \mathbb{R}^n) \equiv (\mathbf{C}^1_c(\mathbb{R}^n))^n$. Then*

$$\text{(GG)} \qquad \int_E \operatorname{div}(g)\, d\mathcal{L}^n = \int_{\partial E} (N_E, g)_{\mathbb{R}^n}\, d\mathcal{H}^{n-1} ,$$

where $N_E : \partial E \to \mathbf{S}^{n-1}$ denotes the (continuous) outward unit normal to E and

$$\operatorname{div}(g)(x) := \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}(x) \quad x \in \mathbb{R}^n .$$

Proof. See, for instance, [Mag, Theorem 9.3]. □

Remark 5.8. Notice that if E is an open set with $\mathbf{C}^1$ regular boundary ∂E, then its outward normal boundary $N_E : \partial E \to \mathbf{S}^{n-1}$ can be extended to a continuous function $\tilde{N}_E : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $|\tilde{N}_E(x)| \leqslant 1$ for each $x \in \mathbb{R}^n$ (see [Mag, 9.2]).

Gauss-Green formula (GG) can be read in the sense of distributions by means of measure theory as follows:

$$\int_{\mathbb{R}^n} \chi_E \operatorname{div}(g)\, d\mathcal{L}^n = -\int_{\mathbb{R}^n} (-N_E, g)_{\mathbb{R}^n}\, d(\mathcal{H}^{n-1} \llcorner \partial E) \quad \forall\, g \in \mathbf{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)\,. \tag{5.26}$$

By (5.26) and measure theory, we can infer the following suggestions in order to define the class of sets of finite perimeter:

- let $g = \varphi\, e_i$ with $\varphi \in \mathbf{C}^1_c(\mathbb{R}^n)$, where $\{e_1, \dots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$, then, by (5.26) it follows that, if $N_E = (N_E^{(1)}, \dots, N_E^{(n)})$,

$$\int_{\mathbb{R}^n} \chi_E\, \partial_i \varphi\, d\mathcal{L}^n = -\int_{\mathbb{R}^n} \varphi\, (-N_E^{(i)})\, d(\mathcal{H}^{n-1} \llcorner \partial E) \quad \forall\, \varphi \in \mathbf{C}^1_c(\mathbb{R}^n),\ i = 1, \dots, n\,,$$

  that is, characteristic function χ_E admits a *weak gradient* $D\chi_E$ in $\mathbb{R}^n$ represented by finite Radon vector measure $\nu_E = -N_E\, \mathcal{H}^{n-1} \llcorner \partial E : \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}^n$;
- by (5.26) it follows that

$$\int_{\mathbb{R}^n} \chi_E \operatorname{div}(g)\, d\mathcal{L}^n \leqslant \mathcal{H}^{n-1}(\partial E)\ \|g\|_\infty \quad \forall\, g \in \mathbf{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)\,, \tag{5.27}$$

  where $\|g\|_\infty := \sup_{\mathbb{R}^n} |g|$. In particular, by the previous inequality and the density of $\mathbf{C}^1_c(\mathbb{R}^n, \mathbb{R}^n) \equiv (\mathbf{C}^1_c(\mathbb{R}^n))^n$ in $((\mathbf{C}^0_c(\mathbb{R}^n))^n, \|\cdot\|_\infty)$, it follows that the linear functional $L_E : \mathbf{C}^1_c(\mathbb{R}^n, \mathbb{R}^n) \equiv (\mathbf{C}^1_c(\mathbb{R}^n))^n \to \mathbb{R}$

$$L_E(g) := \int_{\mathbb{R}^n} \chi_E \operatorname{div}(g)\, d\mathcal{L}^n$$

  can be extended to a linear functional $\tilde{L}_E : (\mathbf{C}^0_c(\mathbb{R}^n))^n \to \mathbb{R}$, which is also continuous according to Definition 1.69.

Taking the previous suggestions into account, we can introduce the space of functions with bounded variation on $\mathbb{R}^n$, which is a bigger class of sets of finite perimeter.

Definition 5.9. Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(i) If $u \in L^1(\Omega)$ we call *variation of u on* Ω the value

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u(x) \operatorname{div} g\, d\mathcal{L}^n : g \in \mathbf{C}^1_c(\Omega, \mathbb{R}^n),\ |g(x)| \leqslant 1 \right\} \in [0, \infty]. \tag{5.28}$$

(ii) We say that $u \in L^1(\Omega)$ has *bounded variation in* Ω if $|Du|(\Omega) < \infty$.

The space $BV(\Omega)$ is the set of functions $u \in L^1(\Omega)$ with bounded variation in Ω. The space $BV_{\text{loc}}(\Omega)$ is the set of functions $u : \Omega \to \overline{\mathbb{R}}$ such that $u|_{\Omega'} \in BV(\Omega')$ for each open set $\Omega' \Subset \Omega$.

Exercise: Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in \mathbf{C}^1(\Omega)$, Then

$$|Du|(\Omega) = \int_\Omega |\nabla u|\, dx\,.$$

(*Hint:* By (GG)

$$\int_\Omega u(x)\mathrm{div} g\, d\mathcal{L}^n = \int_\Omega (g, \nabla u)_{\mathbb{R}^n}\, d\mathcal{L}^n \quad \forall\, g \in \mathbf{C}^1_c(\Omega, \mathbb{R}^n)) .$$

Theorem 5.10 (Structure of BV functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in BV_{\mathrm{loc}}(\Omega)$. Then there exist a unique Radon measure $\mu_u : \mathcal{B}(\Omega) \to [0, \infty]$ and a Borel measurable vector function $w_u = (w_u^{(1)}, \ldots, w_u^{(n)}) : \Omega \to \mathbf{S}^{n-1}$ such that*

$$\mu_u(A) = |Du|(A) \text{ for each open set } A \subset \Omega, \tag{5.29}$$

$$\int_\Omega u\, \mathrm{div} g\, d\mathcal{L}^n = \int_\Omega (w_u, g)_{\mathbb{R}^n}\, d\mu_u, \tag{5.30}$$

for all $g \in \mathbf{C}^1_c(\Omega, \mathbb{R}^n)$. Moreover

$$Du = (D_1, \ldots, D_n) := -(w_u^{(1)}, \ldots, w_u^{(n)})\, \mu_u : \mathcal{B}_{\mathrm{comp}}(\Omega) \to \mathbb{R}^n$$

is a Radon vector measure such that

$$\int_\Omega u\, \partial_i \varphi\, d\mathcal{L}^n = \int_\Omega \varphi\, w_u^{(i)}\, d\mu_u := -\int_\Omega \varphi\, dD_i u \quad \forall\, \varphi \in \mathbf{C}^1_c(\Omega),\ i = 1, \ldots, , n . \tag{5.31}$$

Viceversa if (5.31) *holds for some $u \in L^1(\Omega)$, a Radon measure $\mu \equiv \mu_u$ and functions $w_i \equiv w_u^{(i)} \in L^1_{\mathrm{loc}}(\Omega, \mu)$ $(i = 1, \ldots, n)$, then $u \in BV_{\mathrm{loc}}(\Omega)$ and* (5.29) *and* (5.30) *respectively holds with*

$$\mu_u(B) := \int_B \sqrt{\sum_{i=1}^n w_i^2}\, d\mu \text{ if } B \in \mathcal{B}(\Omega) , \tag{5.32}$$

$$w_u(x) := \begin{cases} \dfrac{(w_1, \ldots, w_n)}{\sqrt{\sum_{i=1}^n w_i^2}}(x) & \text{if } 0 < \sum_{i=1}^n w_i^2(x) < \infty \\ 0 & \text{otherwise} \end{cases} . \tag{5.33}$$

Notation: In the following, if $u \in BV_{\mathrm{loc}}(\Omega)$, since (5.29), we will identify variation $|Du|$ (5.28) with measure μ_u.

Proof. **1st step:** Let us prove there exist a unique Radon measure $\mu_u : \mathcal{B}(\Omega) \to [0, \infty]$ and a Borel vector function $w_u : \Omega \to \mathbf{S}^{n-1}$ such that (5.29) and (5.30) hold.

Let $L_u : \mathbf{C}^1_c(\mathbb{R}^n, \mathbb{R}^n) \equiv (\mathbf{C}^1_c(\mathbb{R}^n))^n \to \mathbb{R}$ be the linear functional

$$L_u(g) := \int_{\mathbb{R}^n} u\, \mathrm{div}(g)\, d\mathcal{L}^n .$$

Arguing as in (5.27), it follows that, for each open set $\Omega' \Subset \Omega$,

$$|L_u(g)| \leqslant |Du|(\Omega')\, \|g\|_\infty \quad \forall\, g \in \mathbf{C}^1_c(\Omega', \mathbb{R}^n) . \tag{5.34}$$

Let $K \subset \Omega$ be a compact. Then there exists an open set Ω' such that

$$K \subset \Omega' \Subset \Omega .$$

Let $g \in (\mathbf{C}^0_c(\Omega))^n$ with $\mathrm{spt}(g) \subset K$ and let

$$g_h(x) := ((g_1 * \varrho_h)(x), \ldots, (g_n * \varrho_h)(x)) \quad \text{if } x \in \Omega , \tag{5.35}$$

where $(\varrho_h)_h$ is a sequence of mollifiers in $\mathbb{R}^n$. Then, it is easy to see that

$$g_h \in \mathbf{C}_c^1(\Omega', \mathbb{R}^n) \quad \text{if } \frac{1}{h} < d(K, \partial\Omega'),$$
$$|g_h(x)| \leqslant \|g\|_\infty \quad \forall\, x \in \Omega',\ h,$$
$$\lim_{h\to\infty} \|g_h - g\|_\infty = 0.$$

By (5.34), sequence $(L(g_h))_h \subset \mathbb{R}$ is a Cauchy sequence. Thus we can extend functional L_u to a linear functional $\tilde{L}_u : (\mathbf{C}_c^0(\Omega))^n \to \mathbb{R}$ as follows

$$\tilde{L}_u(g) := \lim_{h\to\infty} L_u(g_h)$$

and the limit is independent of the choice of the sequence $(g_h)_h$ converging to g. Moreover

$$\sup\left\{\tilde{L}_u(g) : g \in (\mathbf{C}_c^0(\Omega))^n,\ |g| \leqslant 1,\ \operatorname{spt}(g) \subset K\right\} \leqslant |Du|(\Omega') < \infty.$$

Therefore linear functional $\tilde{L}_u : (\mathbf{C}_c^0(\Omega))^n \to \mathbb{R}$ is continuous according to Definition 1.69. Applying the Riesz representation theorem 1.73 and defining

$$\mu_u := \mu_{\tilde{L}_u}, \quad w_u := w_{\tilde{L}_u},$$

(5.29) and (5.30) follow.

2nd step: To prove (5.31), it is sufficient to choose $g := \varphi\, e_i$ $(i = 1, \ldots, n)$ and use (5.30).

3rd step: Suppose that (5.31) holds for some $u \in L^1(\Omega)$, a Radon measure $\mu \equiv \mu_u$ and functions $w_i \equiv w_u^{(i)} \in L^1_{\text{loc}}(\Omega, \mu)$ $(i = 1, \ldots, n)$, then let us prove that $u \in BV_{\text{loc}}(\Omega)$ and (5.29) and (5.30) respectively hold with μ_u and w_u given by, respectively, (5.32) and (5.33). By assumptions, for each $g = (g_1, \ldots, g_n) \in \mathbf{C}_c^1(\Omega, \mathbb{R}^n)$, if $w = (w_1, \ldots, w_n)$,

$$\begin{aligned}\int_\Omega u \operatorname{div} g\, d\mathcal{L}^n &= \int_\Omega u \sum_{i=1}^n \partial_i g_i\, d\mathcal{L}^n = \sum_{i=1}^n \int_\Omega u\, \partial_i g_i\, d\mathcal{L}^n \\ &= \sum_{i=1}^n \int_\Omega w_i\, g_i\, d\mu = \int_\Omega \sum_{i=1}^n w_i\, g_i\, d\mu \\ &= \int_\Omega (w, g)_{\mathbb{R}^n}\, d\mu.\end{aligned} \tag{5.36}$$

Let us fix an open set $\Omega' \Subset \Omega$ and let $g \in \mathbf{C}_c^1(\Omega', \mathbb{R}^n)$. Since $w \in (L^1_{\text{loc}}(\Omega))^n$, it follows that

$$\int_\Omega (w, g)_{\mathbb{R}^n}\, d\mu \leqslant \|g\|_\infty \int_{\Omega'} |w|\, d\mu < \infty. \tag{5.37}$$

Thus, by (5.36) and (5.37), we obtain that

$$|Du|(\Omega') \leqslant \int_{\Omega'} |w|\, d\mu < \infty,$$

for each open set $\Omega' \Subset \Omega$, that is, $u \in BV_{\text{loc}}(\Omega)$. By (5.30) and (5.36), it follows that

$$\int_\Omega (w_u, g)_{\mathbb{R}^n}\, d\mu_u = \int_\Omega (w, g)_{\mathbb{R}^n}\, d\mu, \tag{5.38}$$

for each $g \in \mathbf{C}_c^1(\Omega, \mathbb{R}^n)$. By using the approximation by convolution defined in (5.35), one can prove that (5.38) still holds for each $g \in \mathbf{C}_c^0(\Omega, \mathbb{R}^n)$. The approximation by continuous functions in L^p (see Remark 1.64) enables us to use functions $g = w_u\, \chi_{\Omega'}$ and $g = \dfrac{w}{|w|}\, \chi_{\Omega'}$ in (5.38), for each open set $\Omega' \subset \Omega$, which yields inequalities

$$\mu_u(\Omega') \leqslant \int_{\Omega'} |w|\, d\mu \text{ and } \mu_u(\Omega') \geqslant \int_{\Omega'} |w|\, d\mu\,,$$

that is

$$\mu_u(\Omega') = \int_{\Omega'} |w|\, d\mu \quad \text{ for each open set } \Omega' \subset \Omega\,,$$

Thus (5.32) follows by the approximation in measure of Borel sets by means of open sets from above. By (5.32) and (5.38), (5.33) follows. □

BV functions of one variable ([AFP, Sect. 3.2]) ????.

Definition 5.11. A (Lebesgue) measurable set $E \subset \mathbb{R}^n$ is of *locally finite perimeter* in an open set $\Omega \subset \mathbb{R}^n$ (or is a *Caccioppoli set*) if the characteristic function $\chi_E \in BV_{\text{loc}}(\Omega)$. In this case we call the *perimeter of* E the measure

$$|\partial E| := |D\chi_E| \tag{5.39}$$

and we call the *(generalized inward unit) normal* to ∂E in Ω the vector

$$\nu_E(x) := -w_{\chi_E}(x). \tag{5.40}$$

Remark 5.12. Observe that, by (5.40) and (5.30), it follows that, if $E \subset \mathbb{R}^n$ is a set of locally finite perimeter in an open set $\Omega \subset \mathbb{R}^n$, then it satisfies, in sense of distributions, the following generalized Gauss-Green formula

$$\int_E \operatorname{div}(g)\, d\mathcal{L}^n = -\int_\Omega (\nu_E, g)_{\mathbb{R}^n}\, d|\partial E| \quad \forall\, g \in \mathbf{C}_c^1(\Omega, \mathbb{R}^n)\,. \tag{5.41}$$

Thus vector function $\nu_E : \Omega \to \mathbf{S}^{n-1}$ acts as the inward normal to E in the classical Gauss-Green formula (see (5.26)).

Theorem 5.13. *If $E \subset \mathbb{R}^n$ is an open set with C^1 boundary , then E is a set of locally finite perimeter in $\mathbb{R}^n$ and for each open set $\Omega \subset \mathbb{R}^n$*

$$|\partial E|(\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)\,. \tag{5.42}$$

Proof. By definition of perimeter and (GG)

(5.43)

$$\begin{aligned} |\partial E|(\Omega) &:= \sup\left\{\int_E \operatorname{div} g\, d\mathcal{L}^n : g \in \mathbf{C}_c^1(\Omega, \mathbb{R}^n),\ |g| \leqslant 1\right\} \\ &= \sup\left\{\int_{\partial E \cap \Omega} (N_E, g)_{\mathbb{R}^n}\, d\mathcal{H}^{n-1} : g \in \mathbf{C}_c^1(\Omega, \mathbb{R}^n),\ |g| \leqslant 1\right\} \leqslant \mathcal{H}^{n-1}(\partial E \cap \Omega) \end{aligned}$$

where $N_E : \partial E \to \mathbf{S}^{n-1}$ is the outward unit normal to E. To prove the reverse inequality, let us observe that, by Remark 5.8, there exists a continuous function $\tilde{N}_E : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\tilde{N}_E|_{\partial E} = N_E \text{ and } |\tilde{N}_E(x)| \leqslant 1\, \forall\, x \in \mathbb{R}^n\,. \tag{5.44}$$

Let $\varphi \in \mathbf{C}^1_c(\Omega)$ with $0 \leqslant \varphi \leqslant 1$ and let

$$g_h(x) := \Big((\varphi \tilde{N}_E) * \varrho_h\Big)(x) \quad x \in \mathbb{R}^n\,,$$

where $(\varrho_h)_h$ is a sequence of mollifiers in $\mathbb{R}^n$ and g_h is the convolution product in (5.35). It is easy to see that, by (5.44),

$$g_h \in \mathbf{C}^1_c(\Omega, \mathbb{R}^n) \text{ for } h \text{ large }\,,$$

and

$$g_h \to \varphi \tilde{N}_E \text{ uniformly on } \mathbb{R}^n, \quad |g_h(x)| \leqslant \varphi(x)\; \forall\, x \in \mathbb{R}^n\,.$$

Therefore, for given $\varphi \in \mathbf{C}^1_c(\Omega)$ with $0 \leqslant \varphi \leqslant 1$, by the dominated convergence theorem,

$$\begin{aligned}(5.45)\qquad &\sup\left\{\int_{\partial E\cap\Omega} (N_E, g)_{\mathbb{R}^n}\, d\mathcal{H}^{n-1} : g \in \mathbf{C}^1_c(\Omega, \mathbb{R}^n),\ |g(x)| \leqslant 1\right\}\\ &\geqslant \lim_{h\to\infty} \int_{\partial E\cap\Omega} (N_E, g_h)_{\mathbb{R}^n}\, d\mathcal{H}^{n-1} = \int_{\partial E\cap\Omega} \varphi\, d\mathcal{H}^{n-1} \quad .\end{aligned}$$

By (5.43) and (5.45), taking the supremum on all $\varphi \in \mathbf{C}^1_c(\Omega)$ with $0 \leqslant \varphi \leqslant 1$, it follows that

$$|\partial E|(\Omega) \geqslant \mathcal{H}^{n-1}(\partial E \cap \Omega)\,.$$

□

Remark 5.14. Notice that (5.42) may not hold true if E is a set of finite perimeter but its boundary ∂E is no more regular. Indeed there exists a set $E \subset \mathbb{R}^n$ $(n \geqslant 2)$ of finite perimeter in $\mathbb{R}^n$, that is $|\partial E|(\mathbb{R}^n) < \infty$, but $0 < \mathcal{H}^n(\partial E) = \mathcal{L}^n(\partial E) < \infty$ (see, for instance,[Mag, Example 12.25]). In particular $\mathcal{H}^{n-1}(\partial E) = \infty$. Thus a set of finite perimeter may have a wild topological boundary.

Remark 5.15. The perimeter is invariant under translations, that is

$$(5.46)\qquad |\partial(p+E)|(p+A) = |\partial E|(A), \quad \forall p \in \mathbb{R}^n \text{ and for any Borel set } A \subset \mathbb{R}^n\,,$$

if

$$p + E := \{p + x : x \in E\}\ .$$

Indeed the differential operator div is invariant under translations and the n-dimensional Lebesgue measure $\mathcal{L}^n$ is invariant under traslations. Moreover the perimeter is homogeneous of degree $n-1$ with respect to the dilations , that is

$$(5.47)\qquad |\partial(\lambda E)|(\lambda A) = \lambda^{n-1}|\partial E|(A), \quad \forall\, \lambda > 0 \text{ and for any Borel set } A \subset \mathbb{R}^n\,,$$

if

$$\lambda E := \{\lambda\, x : x \in E\}\ .$$

Also this fact is elementary and can be proved by changing variables in formula (5.28).

Let us recall some simple properties of sets of (locally) finite perimeter,

Proposition 5.16. *L $\Omega \subset \mathbb{R}^n$ be an open set and let E and F be measurable subsets of $\mathbb{R}^n$. Then*

(i) $\mathrm{spt}(|\partial E|) \subset \partial E$*;*
(ii) $|\partial E|(\Omega) = |\partial(\mathbb{R}^n \setminus E)|(\Omega)$*;*

(iii) *(locality of the perimeter measure)* $|\partial E|(\Omega) = |\partial(E \cap \Omega)|(\Omega)$;
(iv) $|\partial(E \cup F)|(\Omega) + |\partial(E \cap F)|(\Omega) \leqslant |\partial E|(\Omega) + |\partial F|(\Omega)$.

Proof. See [AFP, Proposition 3.38]. □

The direct methods of the calculus of variations apply to the sets of finite perimeter and, more generally, to the space of bounded variations since they satisfies the two important properties of semicontinuity and compactness.

Theorem 5.17 (Lower semicontinuity in BV). *Let $u, u_h \in L^1_{\mathrm{loc}}(\Omega)$, $h \in \mathbb{N}$ and suppose that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, that is*

$$u_h \to u \text{ in } L^1(K) \text{ as } h \to \infty, \text{ for each compact set } K \subset \Omega .$$

(i) *Then*

$$\liminf_{h\to\infty} |Du_h|(\Omega) \geqslant |Du|(\Omega) .$$

(ii) *Assume also that*

$$\sup\{|Du_h|(\Omega') : h \in \mathbb{N}\} < \infty \qquad \text{for each open set } \Omega' \Subset \Omega .$$

Then $u \in BV_{\mathrm{loc}}(\Omega)$ and $Du_h \overset{*}{\rightharpoonup} Du$ in Ω, that is,

$$\int_\Omega \varphi \, dDu_h \to \int_\Omega \varphi \, dDu \quad \forall \varphi \in \mathbf{C}^0_c(\Omega) .$$

Proof. See [AFP, Propositions 3.6 and 3.16]. □

Theorem 5.18 (Compactness in BV). *Every sequence $(u_h)_h \subset BV_{\mathrm{loc}}(\Omega)$ satisfying*

$$\sup\left\{\int_{\Omega'} |u_h| dx + |Du_h|(\Omega') : h \in \mathbb{N}\right\} < \infty \text{ for each open set } \Omega' \Subset \Omega , \tag{5.48}$$

admits a subsequence $(u_{h_k})_k$ converging in $L^1_{\mathrm{loc}}(\Omega)$ to $u \in BV_{\mathrm{loc}}(\Omega)$. If Ω is a bounded open set with $\mathbf{C}^1$ boundary, $(u_h)_h \subset BV(\Omega)$ and (5.48) holds with $\Omega' \equiv \Omega$, then $u \in BV(\Omega)$ and

$$u_{h_k} \to u \text{ in } L^1(\Omega) \text{ as } k \to \infty .$$

Proof. See [AFP, Theorem 3.23]. □

The previous results for the space of bounded variations simply yield the following ones for sets of finite perimeter.

Definition 5.19. Given (Lebesgue) measurable sets $(E_h)_h$ and E in $\mathbb{R}^n$ and an open set $\Omega \subset \mathbb{R}^n$,

(i) we say that $(E_h)_h$ *locally converges to E in Ω*, and write $E_h \overset{\mathrm{loc}}{\to} E$ in Ω, if $\chi_{E_h} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$, which amounts to

$$|K \cap (E_h \Delta E)| \to 0 \text{ as } h \to \infty, \text{ for each compact } K \subset \Omega .$$

(ii) We say that $(E_h)_h$ *converges to E in Ω*, and write $E_h \to E$ in Ω, if $\chi_{E_h} \to \chi_E$ in $L^1(\Omega)$, which amounts to

$$|\Omega \cap (E_h \Delta E)| \to 0 \text{ as } h \to \infty .$$

Corollary 5.20 (Lower semicontinuity for sets of finite perimeter). *Let $(E_h)_h$ and E be measurable sets of $\mathbb{R}^n$ and suppose $E_h \overset{\mathrm{loc}}{\to} E$.*

(i) *Then*

$$\liminf_{h\to\infty} |\partial E_h|(\Omega)) \geqslant |\partial E|(\Omega))\,.$$

(ii) *Assume also that*

$$\sup\{|\partial E_h|(\Omega') : \, h \in \mathbb{N}\} < \,\infty \qquad \textit{for each open set } \Omega' \Subset \Omega\,.$$

Then E is a set of locally finite perimeter in Ω and $\nu_{E_h}\,|\partial E_h| \overset{*}{\rightharpoonup} \nu_E\,|\partial E|$ in Ω, that is,

$$\int_\Omega \varphi\,\nu_{E_h}\,d|\partial E_h| \to \int_\Omega \varphi\,\nu_E\,d|\partial E| \quad \forall\,\varphi \in \mathbf{C}^0_c(\Omega)\,.$$

where ν_F denotes the generalized inward normal to F, if F is a set of locally finite perimeter.

Proof. The proof is immediate by Theorem 5.17. □

Corollary 5.21 (Compactness for sets of finite perimeter)**.** *Let $(E_h)_h$ be a sequence of sets with locally finite perimeter in an open set $\Omega \subset \mathbb{R}^n$, satisfying*

(5.49) $$\sup\{|\Omega' \cap E_h| + |\partial E_h|(\Omega') : h \in \mathbb{N}\} < \,\infty \textit{ for each open set } \Omega' \Subset \Omega\,.$$

Then there exist a subsequence $(E_{h_k})_k$ and a set E with locally finite perimer in Ω such that

$$E_h \overset{\text{loc}}{\to} E \;\textit{in}\; \Omega\,.$$

If Ω is a bounded open set with $\mathbf{C}^1$ boundary, $(E_h)_h$ is a sequence of sets with finite perimeter in Ω and (5.49) *holds with $\Omega' \equiv \Omega$, then E is a set with finite perimeter in Ω and*

$$E_{h_k} \to E\,.$$

Proof. The proof easily follows from Theorem 5.18 and the following exercise.

Exercise: If $u_h = \chi_{E_h}$, $u \in L^1_{\text{loc}}(\Omega)$ and

$$\chi_{E_h} \to u \text{ in } L^1_{\text{loc}}(\Omega)\,,$$

then $u = \chi_E$ $\mathcal{L}^n$-a.e. in Ω, for some measurable set $E \subset \Omega$.

□

Minimal boundaries

De Giorgi [DGCP, Theorem 1.1, Chap. II] (see also [G] and [Mag, Proposition 12.29]) proved the existence of minimizers for a Plateau type-problem concerning a geometric variational problem dealing with sets of finite perimeter.

Theorem 5.22 (Existence of minimal boundaries à la De Giorgi, 1960)**.** *Let $A \subset \mathbb{R}^n$ be a bounded set and let $L \subset \mathbb{R}^n$ be a measurable set with $|\partial L|(\mathbb{R}^n) < \infty$. Then there exists a solution of the minimization problem*

(5.50) $$\min\{|\partial F|(\mathbb{R}^n) : \, F \subset \mathbb{R}^n \text{ measurable, } F \setminus A = \, L \setminus A\}$$

Remark 5.23. In some sense the set L determines the boundary value of F. Roughly speaking, prescribing that $F \setminus A = \, L \setminus A$ we impose $L \cap \partial A$ as a "boundary condition" for the admissible sets F. At the same time, the set A, being the region where L can be modified to minimize perimeter, may act as an obstacle. In general, we do not expect uniqueness of minimizers for this problem (see [Mag, Sect. 12.5]).

Proof of Theorem 5.22. **1st step:** Let us denote by

$$X := \{F \subset \mathbb{R}^n : F \text{ has locally finite perimeter in } \mathbb{R}^n,\ F \setminus A = L \setminus A\}$$

Since $L \in X$, minimization problem (5.50) is equivalent to the problem

$$\min\{|\partial F|(\mathbb{R}^n) : F \in X\} \tag{5.51}$$

2nd step: We are going to apply the direct methods of the calculus of variations for showing the existence of problem (5.51). More precisely we will apply generalized Weierstrass theorem 5.1 to functional

$$P : X \to [0, \infty], \quad P(F) := |\partial F|(\mathbb{R}^n)$$

and we endow the class of competitors X by topology $\tau = \mathrm{loc}$ of local convergence of sets in $\mathbb{R}^n$ introduced in Definition 5.19 (i). Thus we have to prove that

$$P \text{ is sequentially lower semicontinuous ;} \tag{5.52}$$

$$P \text{ is sequentially coercive.} \tag{5.53}$$

Property (5.52) immediately follows by the lower semicontinuity for sets of finite perimeter (see Corollary 5.20 (i)). Let us prove (5.53). Fix $t \in (0, \infty)$ and let

$$(E_h)_h \subset \{F \in X : P(F) \leqslant t\}\ . \tag{5.54}$$

We have to prove there exist a subsequence $(E_{h_k})_k$ and a set $E \in X$ such that

$$E_{h_k} \overset{\mathrm{loc}}{\to} E \text{ as } k \to \infty\,. \tag{5.55}$$

Observe that

$$E_h = (E_h \cap A) \cup (E_h \setminus A) \subseteq A \cup L \quad \forall\, h\,.$$

This implies that, for each open set $\Omega' \Subset \mathbb{R}^n$

$$|E_h \cap \Omega'| \leqslant |(A \cup L) \cap \Omega'| \quad \forall\, h\,. \tag{5.56}$$

By (5.54) and (5.56), it follows that (5.49) is fulfilled with $\Omega = \mathbb{R}^n$. Thus, by the compactness of sets of finite perimeter (see Corollary 3.11), (5.55) follows. □

IN PROGRESS!

References

[A] L. Ambrosio, *Corso introduttivo alla Teoria Geometrica della Misura ed alle Superfici minime*,Appunti dei corsi tenuti da docenti della Scuola, Scuola Normale Siperiore, Pisa,1997.

[A2] L. Ambrosio, *La teoria dei perimetri di Caccioppoli-De Giorgi e i suoi più recenti sviluppi.* Rend. Lincei Mat. Appl. **21** (2010), 275–286.

[AFP] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Oxford University Press, 2000.

[AG] L. Ambrosio, R. Ghezzi, *Sobolev and bounded variation functions on metric measure spaces*, Geometry, Analysis and Dynamics on sub-Riemannian manifolds, vol. II, D. Barilari, U. Boscain, M. Sigalotti edts., EMS Series of Lecture Notes in Mathematics, 2016.

[AK] L. Ambrosio, B. Kirchheim, *Rectifiable sets in metric and Banach spaces.* Math. Ann. **318**, (2000), 527-555.

[AT] L. Ambrosio, P. Tilli, *Topics on analysis in metric spaces.* Oxford University Press,Oxford, 2004.

[Ba] S. Banach, *Sur un theorème de M. Vitali*, Fund. Math. **5** (1924), 130–136.

[Be1] A.S. Besicovitch, *A general form of the covering principle and relative differentiation of additive functions*, Proc.Cambridge Philos.Soc. **41** (1945), 103–110.

[Be2] A.S. Besicovitch, *A general form of the covering principle and relative differentiation of additive functions II*, Proc. Cambridge Philos. Soc.**41** (1946), 1–10.

[Be3] A.S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Ann. **98** (1928), 422–464.

[Be4] A.S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points II*, Math. Ann. **115** (1938), 296–329.

[Be5] A.S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points III*, Math. Ann. **116** (1939), 349–357.

[B] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2011. An Italian version is also available: H. Brezis, *Analisi Funzionale- Teoria e applicazioni*, Liguori, Napoli, 1986.

[C] C. Carathéodory , *Vorlesungen uber reelle Funktionen*, Leipzig, 1927.

[C2] C. Carathéodory,*Über das lineare Mass von Punktmengen, eine Verallgemeinerung des Längenbegriffs*, Nach. Ges. Wiss. Gottingen (1914), 406–426.

[Ch] J. Cheeger *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Func. Anal. **9** (1999),428–517.

[Co] D. L. Cohn, *Measure Theory*, Birkhäuser, 1980.

[DG1] E. De Giorgi, *Su una teoria generale della misura* $(r-1)$*-dimensionale in uno spazio ad* r *dimensioni.* Ann.Mat.Pura Appl.(4), **36**, (1954), 191–213.

[DG2] E. De Giorgi, *Nuovi teoremi relativi alle misure* $(r-1)$*-dimensionali in uno spazio ad* r *dimensioni*,Ricerche Mat. **4**, (1955), 95–113.

[DG3] E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera nella classe degli insiemi aventi frontiera orientata di misura finita*, Memorie Acc. Naz. Lincei Ser. VIII **5**, (1958), 33–44.

[DGCP] E. De Giorgi, F. Colombini, L.C. Piccinini, *Frontiere orientate di misura minima e questioni collegate.* Scuola Normale Superiore, Pisa, 1972.

[dG] M. de Guzmán, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Math., Springer-Vrerlag, 1975.

[EG] L.C. Evans, R. Gariepy, *Lecture Notes on Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[Fa] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, Second Edition,Chichester, John Wiley and Sons, 2003.

[Fe] H. Federer, *Geometric Measure Theory*, Springer, 1969.

[F] G.B. Folland , *Real Analysis. Modern techniques and their applications*, Second Edition, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[GZ] R. F. Gariepy, W. P. Ziemer, *Modern Real Analysis*, PWS Publishing Company, Boston, 1994.

[GO] B. R. Gelbaum, J. M. H. Olmsted *Counterexamples in Analysis*, Dover Publications,Inc., New York, 2003.

[GH] M. Giaquinta, S. Hildebrandt, *Calculus of Variations I*,Springer, Berlin Heidelberg, 2004.

[G] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston, 1984.

[G2] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific, Singapore, 2003.

[HOH] H. Hanche-Olsen, H. Holge, *The Kolmogorov-Riesz compactness theorem*, Expo. Math. **28** (2010), no. 4, 385–394.

[Ha] F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann. **79** (1919), 157–179.

[He] J. Heinonen *Lectures on analysis on metric spaces*, Springer, New York, 2001.

[He2] J. Heinonen *Lectures on Lipschitz analysis*, Report. University of Jyväskylä Department of Mathematics and Statistics, **100**. University of Jyväskylä, Jyväskylä, 2005. ii+77 pp.

[LeR] E. Le Donne, S. Rigot, *Besicovitch covering property on graded groups and applications to measure differentiation*, Preprint 2015.

[Le] H. Lebesgue, *Lecons sur l'integration et la recherche de fonctions primitives*, Deuxieme Edition, Gauthier-Villars, Paris, 1926.

[Le2] H. Lebesgue, *Sur l'intégration des fonctions discontinues*, Ann. Sci. École Norm. Sup. (3) **27** (1910), 361–450.

[Mag] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems, An Introduction to Geometric Measure Theory* , Cambridge University Press, 2012.

[MM] U. Massari, M. Miranda *Minimal Surfaces of Codimension One*, North-Holland, Amsterdam, 1984.

[Ma] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[McS] E.J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), 837–842.

[Mi] M. Miranda, *Un Teorema di esistenza e unicità per il problema dell'area minima in n variabili.* Ann.Sc.Norm.Sup. Pisa III, **19** (1965), 233–249.

[Mo] F. Morgan, *Geometric Measure Theory, A beginner's Guide*, Academic Press, 1988.

[Mor] A.P. Morse, *Perfect blankets*, Trans. Amer. Math. Soc. **6** (1947), 418–442.

[Ni] O. Nikodym, *Sur une génésalization des intégrales de M. J. Radon*, Fund. Math. **15** (1930), 131–179.

[Pe] J. Pérez, *A New Golden Age of Minimal Surfaces*, Notices of the AMS **64** (2017), 347–358.

[P] A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007.

[Pre] D. Preiss, *Dimension of metrics and differentiation of measures*, General topology and its relations to modern analysis and algebra, V (Prague, 1981), Sigma Ser. Pure Math., vol. 3, Heldermann, Berlin, 1983, 565–568.

[Pre2] D. Preiss, *Geometry of measures in* $\mathbb{R}^n$*: distributions. rectifiability, and densities*, Ann. of Math. **125** (1987), 537–643.

[Rad] H. Rademacher,*Uber partielle und totale Differenziarbarkeit I*, Math. Ann. **79** (1919), 340–359.

[Ra] J. Radon, *Theorie und Anwendungen der Theorie der absolut additiven Mengenfunktionen*, Sitzungsber. Kaiserl. (Österreich) Akad. Wiss. Math.- Nat. Kl. , Abtteilung IIa **122**, (1913), 1295–1438.

[Ro] H. L. Royden, *Real Analysis*, Macmillan Publishing Company, New York, 1988.

[R1] W. Rudin, *Real and Complex Analysis*, Third Edition, McGraw-Hill Book Co.,New York,1987.

[R2] W. Rudin, *Functional Analysis*, McGraw-Hill Book Co.,New York,1973.

[Sch] A.R. Schep, *Addendum to "A still one more proof of the Radon-Nikodym Theorem"*, 2006, available at www.math.sc.edu/ schep/Radon-update.pdf

[Ser] R. Serapioni *Notes for the course in Geometric Measure Theory-Master degree in Mathematics-University of Trento.*

[SC] F. Serra Cassano *Notes of Advanced Analysis-a.y. 2016/17-Master degree in Mathematics-University of Trento.*

[SC2] F. Serra Cassano, *Some topics of geometric measure theory in Carnot groups*, Geometry, Analysis and Dynamics on sub-Riemannian manifolds, vol. I, D. Barilari, U. Boscain, M. Sigalotti edts., EMS Series of Lecture Notes in Mathematics, 2016.

[So] S.L. Sobolev, *On a theorem of functional analysis* (English) , Amer. Math. Soc. Transl. (2) **34** (1963), 39–68.

[Ti] J. Tišer, *Vitali covering theorem in Hilbert space*, Trans. Amer. Math. Soc. **355** (2003), 3277–3289.

[T] F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press,New York,1967.

[Vitali] G. Vitali, *Sulle funzioni integrabili*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. **40** (1904-05), 1021–1034.

[Vitali2] G. Vitali, *Sui gruppi di punti e sulle funzioni di variabili reali*, Atti Accad. Sci. Torino **43** (1908), 75–92.

INFORMATION ABOUT SOME QUOTED MATHEMATICIANS

Biographical and scientific information more detailed may find at the website http:///www-history.mcs.st-andrews.ac.uk/

- ALAOGLU Leonidas (1914, Red Deer, Alberta, Canada-1981,): Alaoglu was a Canadian-American mathematician, most famous for his widely-cited result called Alaoglu's theorem on the weak-star compactness of the closed unit ball in the dual of a normed space, also known as the Banach-Bourbaki-Alaoglu theorem.
- BAIRE René-Louis (1874, Paris, France- 1932, Chambéry, France): Baire worked on the theory of functions and the concept of a limit. He is best known for the Baire category theorem, a result he proved in his 1899 thesis.
- BANACH Stefan (1892, Kraków, Austria-Hungary (now Poland) - 1945 in Lvov, (now Ukraine)): Banach founded modern functional analysis and made major contributions to the theory of topological vector spaces. In addition, he contributed to measure theory, integration, and orthogonal series.
- BESICOVITCH Abram S. (1891, Berdyansk, Russia - 1970, Cambridge, England): He was a Russian mathematician, who worked mainly in England. He worked mainly on combinatorial methods and questions in real analysis, and gave important contributions in topics such as the Kakeya needle problem, the Hausdorff-Besicovitch dimension and the rectifiability in the plane.
- BOREL Emil F.E.J. (1871, Saint Affrique, Aveyron, Midi-Pyrénées, France - 1956, Paris): Borel created the first effective theory of the measure of sets of points, beginning of the modern theory of functions of a real variable.
- BOURBAKI Nicolas: Nicolas Bourbaki is the pseudonym of a group of (mainly) French mathematicians who published an authoritative account of contemporary mathematics.
- CACCIOPPOLI Renato (1904, Napoli- 1959, Napoli) was an Italian mathematician, known for his deep contributions to mathematical analysis, including the theory of functions of several complex variables, functional analysis, measure theory and partial differential equations. In particular he pioneered some issues of geometric measure theory by means of the introduction of some sets today called *Caccioppoli's sets*.
- CANTOR George F.L.P. (1845, St Petersburg, Russia - 1918, Halle, Germany): Cantor founded the set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers. He also advanced the study of trigonometric series.
- CARATHÉODORY Constantin (1873, Berlin - 1950, Munich): Carathéodory made significant contributions to the calculus of variations, the theory of point set measure, and the theory of functions of a real variable.
- CAUCHY Augustin-Louis (1789, Paris, France - 1857, Sceaux (near Paris), France) Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.
- DE GIORGI Ennio (1928, Lecce, Italy - 1996, Pisa, Italy): he was an Italian mathematician, who worked on calculus of variations, geometric measure theory, partial differential equations and the foundations of mathematics, giving fundamental

contributions. In particular he did the last step for solving the 19th Hilbert problem on the regularity of solutions of elliptic partial differential equations (togheter with J. F. Nash, but independently) and solved the so-called Bernstein problem for minimal surfaces (in collaboration with E. Bombieri and E. Giusti).

• DE LA VALLÉE POUSSIN Charles J. (1866, Louvain, Belgio-1962, Louvain, Belgio): he was a Belgian mathematician and is best known for proving the prime number theorem.

• DIRICHLET Gustav L. (1805, Düren, French Empire (now Germany)- 1859, Göttingen, Hanover (now Germany)) He made valuable contributions to number theory, analysis, and mechanics. In number theory he proved the existence of an infinite number of primes in any arithmetic series. In mechanics he investigated the equilibrium of systems and potential theory, which led him to the Dirichlet problem concerning harmonic functions with prescribed boundary values.

• EGOROFF Dimitri F. (1869, Moscow, Russia - 1931, Kazan, USSR): He worked on integral equations and a theorem in the theory of functions of real variable is named after him. Luzin was Egorov's first student and became a member of the school Egorov created in Moscow dealing with functions of real variable.

• EULER Leonhard (1707, Basel, Switzerland -1783, St Petersburg, Russia) He was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.He is also renowed for his work in mechanics, fluid dynamics, optics, and astronomy.

• FATOU Pierre J.L. (1878, Lorient, France - 1929, Pornichet, France): Fatou worked in the fields of complex analytic dynamic and iterative and recursive processes.

• FISHER Ernst (1875, Vienna, Austria - 1954, Cologne, Germany): Ernst Fischer is best known for the Riesz-Fischer theorem in the theory of Lebesgue integration.

• FOURIER Joseph J.B. (1768, Auxerre, Francia - 1830, Parigi): Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

• FRÉCHÉT Maurice (1878, Maligny - 1973, Paris) Fréchét was a French mathematician who made major contributions to the topology of point sets and defined and founded the theory of abstract spaces.In particular, in his thesis he introduced the concept of a metric space, although he did not invent the name 'metric space' which is due to Hausdorff.

• FRIEDRICHS Kurt Otto (1901, Kiel, Germany- 1982, New Rochelle, New York, USA) Friedrichs' greatest contribution to applied mathematics was his work on partial differential equations. He also did major research and wrote many books.

• FUBINI Guido, (1879, Venice, Italy -1943, New York, USA) Fubini may be considered one of the founder of modern projective-differential geometry. He has been very important as an analyst (Lebesgue integrals) and in the study of automorphic functions and discontinuous groups. He has been also engaged in mathematical physics and in applied mathematics.

• GAUSS Carl F. (1777, Brunswick, Duchy of Brunswick (now Germany) - 1855, Göttingen, Hanover (now Germany)) Gauss worked in a wide variety of fields in both mathematics and physics incuding number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. His work has had an immense influence in many areas.

• GREEN George (1793, Sneinton, Nottingham, England - 1841, Sneinton, Nottingham, England) George Green was an English mathematician best-known for Green's function and Greeen's theorems in potential theory.

• HADAMARD Jacques S. (1865, Versailles, Francia - 1963, Parigi)???

• HAHN Hans (1879, Vienna, Austria - 1934, Vienna, Austria): Hahn was an Austrian mathematician who is best remembered for the Hahn-Banach theorem. He also made important contributions to the calculus of variations, developing ideas of Weierstrass.

• HAUSDORFF Felix (1868, Breslau, Germany (now Wroclaw, Poland)- 1942, Bonn, Germany): Hausdorff worked in topology creating a theory of topological and metric spaces. In particular, he introduced the modern notion of metric space. He also worked in set theory and introduced the concept of a partially ordered set.

• HILBERT David (1862, Königsberg, Prussia (now Kaliningrad, Russia)-1943, Göttingen, Germany): Hilbert's work in geometry had the greatest influence in that area after Euclid. A systematic study of the axioms of Euclidean geometry led Hilbert to propose 21 such axioms and he analysed their significance. He made contributions in many areas of mathematics and physics.

• HÖLDER Otto L. (1859, Stuttgart, Germany - 1937, Leipzig, Germany): Hölder worked on the convergence of Fourier series and in 1884 he discovered the inequality now named after him. He became interested in group theory through Kronecker and Klein and proved the uniqueness of the factor groups in a composition series.

• KOLMOGOROV Andrey N. (1903, Tambov, Tambov province, Russia - 1987, Moscow) He was a Soviet Russian mathematician, preeminent in the 20th century, who advanced various scientific fields, among them probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity.

• JORDAN Camille M.E. (1838, La Croix-Rousse, Lyon, France - 1922, Paris, France): Jordan was highly regarded by his contemporaries for his work in algebra, group theory and Galois theory. Jordan is best remembered today among analysts and topologists for his proof that a simply closed curve divides a plane into exactly two regions, now called the Jordan curve theorem. He also originated the concept of functions of bounded variation and is known especially for his definition of the length of a curve.

• LAGRANGE Joseph-Louis (1736, Turin, Sardinia-Piedmont (now Italy) - 1813 in Paris, France) Born Giuseppe Lodovico (Luigi) Lagrangia, he was a mathematician and astronomer, lived part of his life in Prussia and part in France, making great contributions to all fields of analysis, to number theory, and to classical and celestial mechanics. In particular he was one of the founders of the caculus of variations.

• LEBESGUE Henry L. (1875, Beauvais, Oise, Picardie, France-1941, Paris, France): Lebesgue formulated the theory of measure in 1901 and the following year he gave the definition of the Lebesgue integral that generalises the notion of the Riemann integral.

• LEVI Beppo (1875, Turin, Italy - 1961, Rosarno, Argentina): He studied singularities on algebraic curves and surfaces. Later he proved some foundational results concerning Lebesgue integration.

• LIPSCHITZ Rudolf O.S.(1832, Könisberg, Germany (now Kaliningrad, Russia) -1903, Bonn, Germany) He was a German mathematician and professor at the University of Bonn from 1864. Dirichlet was his teacher. While Lipschitz gave his name to the Lipschitz continuity condition, he worked in a broad range of areas. These included number theory, algebras with involution, mathematical analysis, differential geometry and classical mechanics.

• LUSIN Nikolai N. (1883, Irkutsk, Russia - 1950, Moscow, USSR): Lusin's main contributions are in the area of foundations of mathematics and measure theory. He also made significant contributions to descriptive set topology.

• MINKOWSKI Hermann (1864, Alexotas, Russian Empire (now Kaunas, Lithuania) - 1909, Göttingen, Germany): Minkowski developed a new view of space and time and laid the mathematical foundation of the theory of relativity.

• MORSE Anthony P. (1911-1984): he was an American mathematician who worked in both analysis, especially measure theory, and in the foundations of mathematics. He is best known as the co-creator, together with John L. Kelley, of Morse-Kelley set theory. This theory first appeared in print in Kelley's General Topology. He is also known for his work on the Morse-Sard theorem and the Federer-Morse theorem.

• NIKODYM Otto M. (1887, Zablotow, Galicia, Austria-Hungary (now Ukraine) - 1974, Utica, USA): Nikodym's name is mostly known in measure theory (e. g. the Radon-Nikodym theorem and derivative, the Nikodym convergence theorem, the Nikodym-Grothendieck boundedness theorem), in functional analysis (the Radon-Nikodym property of a Banach space, the Frechet-Nikodym metric space, a Nikodym set), projections onto convex sets with applications to Dirichlet problem, generalized solutions of differential equations, descriptive set theory and the foundations of quantum mechanics.

• PEANO Giuseppe (1858, Cuneo, Italy - 1932, Turin, Italy): Peano was the founder of symbolic logic and his interests centred on the foundations of mathematics and on the development of a formal logical language. Among his important contributions, let us recall he invented 'space-filling' curves in 1890, these are continuous surjective mappings from [0,1] onto the unit square.

• PLATEAU Joseph A. F. (Tournal, Belgium, 1801- Ghent, Belgium, 1883): he was a Belgian physicist, who studied the phenomena of capillary action and surface tension. The mathematical problem of existence of a minimal surface with a given boundary is named after him *Plateau's problem.* He conducted extensive studies of soap films and formulated Plateau's laws which describe the structures formed by such films in foams.

• POINCARÉ Jules Henri (1854, Nancy, Meurthe-et-Moselle - 1912, Paris) He was a French mathematician, theoretical physicist, and a philosopher of science. He is often described as a polymath, and in mathematics as The Last Universalist, since he excelled in all fields of the discipline as it existed during his lifetime. As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He is considered to be one of the founders of the field of topology.

• POISSON S. Denis (1781, Pithiviers, France - 1840, Sceaux near Paris) He was very well-known for his work on definite integrals, electromagnetic theory, and probability. PoissonÕs most important work concerned the application of mathematics to electricity and magnetism, mechanics, and other areas of physics. Poisson contributed to celestial mechanics by extending the work of Lagrange and Laplace on the stability of planetary orbits and by calculating the gravitational attraction exerted by spheroidal and ellipsoidal bodies. He also did important investigation of probability. In pure mathematics his most important works were a series of papers on definite integrals and his advances in Fourier analysis, which paved the way for the research of the German mathematicians Peter Dirichlet and Bernhard Riemann.

• PRYM Friedrich E.F. (1841 Düren - 1915 Bonn) He was a German mathematician who introduced Prym varieties and Prym differentials.

• RADEMACHER Hans (1892, Wandsbeck, now Hamburg-Wandsbek - 1969, Haverford, Pennsylvania, USA) Rademacher performed research in analytic number theory, mathematical genetics, the theory of functions of a real variable, and quantum theory. Most notably, he developed the theory of Dedekind sums. Rademacher's name is also known for the result about the differentiability of Lipschitz functions.

• RADON Johann (1887, Tetschen, Bohemia (now Decin, Czech Republic) - 1956, Vienna, Austria): Radon worked on the calculus of variations, differential geometry and measure theory.

• RIEMANN G. F. Bernhard (1826, Breselenz, Hanover (now Germany)- 1866, Selasca, Italy): Riemann's ideas concerning geometry of space had a profound effect on the development of modern theoretical physics. He clarified the notion of integral by defining what we now call the Riemann integral.

• RIESZ Frigyes (Friedrich) (1880, Györ, Austria-Hungary (now Hungary) - 1956, Budapest, Hungary): Riesz was a founder of functional analysis and his work has many important applications in physics.

• RIESZ Marcel (1886, Györ, Austria-Hungary (now Hungary) - 1969, Lund, Sweeden) He was a Hungarian mathematician and moved to Sweden in 1908 and spent the rest of his life there. He was known for work on classical analysis, on fundamental solutions of partial differential equations, on divergent series, Clifford algebras, and number theory. He was the younger brother of the mathematician Frigyes Riesz.

• SCHAUDER Juliusz P. (1899, Lemberg, Austrian Empire (now Lviv, Ukraine) - 1943, Lwów, Poland (now Ukraine)) He was a Polish mathematician of Jewish origin, known for his fundamental work in functional analysis, partial differential equation and mathematical physics. Schauder was Jewish, and after the invasion of German troops in Lwów it was impossible for him to continue his work. He was executed by the Gestapo, probably in September 1943.

• SERRIN James (1926, Chicago, Illinois, USA - 2012, Minneapolis, Minnesota, USA) Serrin is a mathematician well-known for his contributions to continuum mechanics, nonlinear analysis, and partial differential equations.

• SOBOLEV Sergei L. (1908, S.Petersburg - 1989, Moscow): He introduced the notions that are now fundamental for several different areas of mathematics. Sobolev spaces and their embedding theorems are an important subject in functional analysis.

• STEINHAUS Hugo D. (1887, Jaslo, Galicia, Austrian Empire (now Poland) - 1972, Wroclaw, Poland): He did important work on functional analysis. Some of Steinhaus's early work was on trigonometric series. He was the first to give some examples which would lead to marked progress in the subject.

• TONELLI Leonida (1885, Gallipoli, Italy - 1946, Pisa, Italy) Tonelli discovered the importance of the semicontinuity in calculus of variations in order to get the existence of minima or maxima for functionals. He also advanced the study of the integration theory.

• URYSOHN Pavel S. (1898, Odessa, Ukraine- 1924, Batz-sur-Mer, France): Urysohn is best known for his contributions in the theory of dimension, and for Urysohn's Metrization Theorem and Urysohn's Lemma, both of which are fundamental results in topology.

• VITALI Giuseppe (1875, Ravenna, Italy - 1932, Bologna, Italy): Vitali made significant mathematical discoveries including a theorem on set-covering, the notion and the characterization of an absolutely continuous functions and a criterion for the closure of a system of orthogonal functions.

• VOLTERRA Vito (1860, Ancona, Italy - 1940, Roma, Italy) Volterra was an Italian mathematician and physicist, known for his contributions to mathematical biology and integral equations, being one of the founders of functional analysis. Volterra is the one among few people who was a plenary speaker in the International Congress of Mathematicians four times (1900, 1908, 1920, 1928). In 1922, he joined the opposition to the Fascist regime of Benito Mussolini and in 1931 he was one of only 12 out of 1,250 professors who refused to take a mandatory oath of loyalty. As a result of his refusal to sign the oath of allegiance to the fascist government, he was compelled to resign his university post and his membership of scientific academies, and, during the following years, he lived largely abroad, returning to Rome just before his death.

• VON NEUMANN John (1903, Budapest, Hungary - 1957, Washington D.C., USA): Von Neumann was generally regarded as the foremost mathematician of his time and said to be "the last representative of the great mathematicians";a genius who was comfortable integrating both pure and applied sciences. He made major contributions to a number of fields, including mathematics (foundations of mathematics, functional analysis, ergodic theory, representation theory, operator algebras, geometry, topology, and numerical analysis), physics (quantum mechanics, hydrodynamics, and quantum statistical mechanics), economics (game theory), computing (Von Neumann architecture, linear programming, self-replicating machines, stochastic computing), and statistics.He was a pioneer of the application of operator theory to quantum mechanics in the development of functional analysis, and a key figure in the development of game theory and the concepts of cellular automata, the universal constructor and the digital computer.

• WEIERSTRASS Karl (1815, Ostenfelde, Germania -1897, Berlino): Weierstrass is best known for his construction of the theory of complex functions by means of power series. Known as the father of modern analysis, Weierstrass devised tests for the convergence of series and contributed to the theory of periodic functions, functions of real variables, elliptic functions, Abelian functions, converging infinite products, and the calculus of variations.