GEOMETRIC MEASURE THEORY
Academic Year 2018/19

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Contents

1. Recalls and complements of measure theory.

1.1. **Measures and outer measures, approximation of measures.**

Recalls of some important notions and results of abstract measure theory concerning outer measures and measures.

Approximation of Radon measures on l.c.s. metric spaces (∘): let (X, d) be a separable, locally compact metric space and φ (respectively μ) be a Radon outer measure (respectively Radon measure) on X. Then

(i) for each $B \subset X$, $\varphi(B) = \inf\{\varphi(U) : U \supset B, U \text{ open}\}$

(respectively, for each $B \in \mathcal{B}(X)$, $\mu(B) = \inf\{\mu(U) : U \supset B, U \,\text{open}\}$);

(ii) for each $B \in \mathcal{M}_\varphi$, $\varphi(B) = \sup\{\varphi(K) : K \subset B, K \,\text{compact}\}$
(respectively, for each $B \in \mathcal{B}(X)$, $\mu(B) = \sup\{\mu(K) : K \subset B, K \,\text{compact}\}$).

1.2. **Convergence and approximation of measurable functions: Severini-Egoroff's and Lusin's theorems.**

Severini-Egoroff's theorem :let $(X, \mathcal{M}, \mu)$ be a measure space with μ finite. Suppose $f_h : X \to \overline{\mathbb{R}}$ $(h = 1, 2, \dots)$ and $f : X \to \overline{\mathbb{R}}$ are measurable functions that are finite μ-a.e.on X. Also, suppose that $(f_h)_h$ converges pointwise μ-a.e. to f. Then for each $\epsilon > 0$ there exists a set $A \in \mathcal{M}$ such that $\mu(X \setminus A) < \epsilon$ and $f_h \to f$ uniformly on A, that is

$$\sup_{x \in A} |f_h(x) - f(x)| \to 0 \text{ as } h \to \infty\,.$$

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Lusin's theorem let μ be a Radon outer measure on a locally compact, separable metric space X . Let $f : X \to \overline{\mathbb{R}}$ be a μ-measurable function such that there exists a Borel set $A \subset X$ with

$$\mu(A) < \infty,\ f(x) = 0 \quad \forall\, x \in X \setminus A \text{ and } |f(x)| < \infty \quad \mu - \text{a.e. } x \in X\,.$$

Then, for each $\epsilon > 0$, there exists $g \in \mathbf{C}^0_c(X)$ such that

$$\mu\left(\{x \in X :\ f(x) \neq g(x)\}\right) < \epsilon\,.$$

Moreover g can be chosen such that

$$\sup_{x \in X} |g(x)| \leqslant \sup_{x \in X} |f(x)|\,.$$

A consequence of Lusin's theorem is the following. Corollary ($\circ$): Let μ be a Radon outer measure on a locally compact, separable metric space X and let $f : X \to \mathbb{R}$ be a μ-measurable function. Then there exist a Borel function $g : X \to \mathbb{R}$ such that $f = g\, \mu$-a.e. on X.

1.3. **Absolutely continuous and singular measures. Radon-Nikodym and Lebesgue decomposition theorems.**

Definitions of absolutely continuous and mutually singular measures in a measure space $(X, \mathcal{M})$.

Recalls of the Radon-Nikodym and Lebesgue's decomposition theorems.

1.4. **Signed vector measures.**

Definition and notions concerning signed measures.

Examples of signed measures.

Lebesgue decomposition theorem for signed measures : Let $(X, \mathcal{M}, \mu)$ be a measure space with μ σ-finite, and $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ be a σ-finite signed measure . Then there are two signed measures ν_{ac}, $\nu_s : \mathcal{M} \to \overline{\mathbb{R}}$ such that

$$\text{(LD)} \qquad \nu_{ac} << \mu, \quad \nu_s \perp \mu, \quad \nu = \nu_{ac} + \nu_s\,,$$

and there exists a measurable function $w : X \to \overline{\mathbb{R}}$ such that either w_+ or w_- is integrable with respect to μ such that

$$\text{(RN)} \qquad \nu_{ac}(E) = \int_E w\, d\mu \quad \forall\, E \in \mathcal{M}\,.$$

Moreover both decomposition (LD) and representation (RN) are unique.

Definitions and notions concerning vector signed measures.

Theorem (Properties of the total variation of a signed vector measure) :

(i) Let ν be a signed vector measure on $(X,\mathcal{M})$. Then its total variation $|\nu|$ is a positive measure.
(ii) If ν is a vector measure, then $|\nu|$ is a positive finite measure, that is $|\nu|(X) < \infty$.

Remark: the above theorem shows that for any real measure ν, its positive and negative part are positive finite measures, hence the decomposition $\nu = \nu^+ - \nu^-$ holds; it is known as the *Jordan decomposition* of ν.

Remark: It is immediate to check that $\mathbb{R}^m$-valued vector measures can be added and multiplied by real numbers, hence they form a real vector space; moreover, an easy consequence of the above theorem is that the total variation is a norm on the space of measures, which turns out to be a Banach space .

Example: Given a measure space $(X,\mathcal{M},\mu)$ and a vector function $w = (w_1,\dots,w_m) : X \to \overline{\mathbb{R}}^m$, with each $w_i : X \to \overline{\mathbb{R}}$ $(i = 1,\dots,m)$ measurable functions such that either $w_{i,+}$ or $w_{i,-}$ is integrable. Let us define the vector set function $\mu_w : \mathcal{M} \to \overline{\mathbb{R}}^m$ defined as follows

$$\mu_w(E) = \int_E w\,d\mu := \left(\int_E w_1\,d\mu,\dots,\int_E w_m\,d\mu\right) \quad E \in \mathcal{M}\,. \tag{1.1}$$

Then it is easy to see that μ_w is a signed vector measure and its total variation is computed in the following proposition.

Proposition ($\circ$) Let $(X,\mathcal{M},\mu)$ be a measure space and let $w = (w_1,\dots,w_m) : X \to \overline{\mathbb{R}}^m$, with each $w_i : X \to \overline{\mathbb{R}}$ $(i = 1,\dots,m)$ measurable functions such that either $w_{i,+}$ or $w_{i,-}$ is integrable. Consider the vector signed measure μ_w in (1.1).

Then

$$|\mu_w|(E) = \int_E |w|\,d\mu \quad \forall E \in \mathcal{M}\,. \tag{1.2}$$

Definitions of integrals on a measure space $(X,\mathcal{M})$ and with respect to a $\mathbb{R}^m$-vector measure.

Definitions of absolute continuity and singularity for vector signed measures.

Lebegue decomposition theorem for vector signed measures ($\circ$): let ν and μ be respectively a $\overline{\mathbb{R}}^m$-valued σ-finite measure and a σ-finite positive measure on a measure space $(X,\mathcal{M})$. Then there is a decomposition of ν such that

$$\nu = \nu_{ac} + \nu_s\,, \tag{1.3}$$

where ν_{ac} and ν_s are still $\overline{\mathbb{R}}^m$-valued signed measures on $(X,\mathcal{M})$ with $\nu_{ac} << \mu$ and $\nu_s \perp \mu$. The decomposition is unique. Moreover there exists a unique vector function $w = (w_1,\dots,w_m) : X \to \overline{\mathbb{R}}^m$ with either $w_{i,+}$ or $w_{i,-}$ $(i = 1,\dots,m)$ integrable functions w.r.t. μ such that

$$\nu_{ac}(E) = \mu_w(E) = \int_E w\,d\mu \quad \forall E \in \mathcal{M}\,. \tag{1.4}$$

Corollary (Polar decomposition for vector measures): let ν be a $\mathbb{R}^m$ -valued measure on the measure space $(X, \mathcal{M})$. Then there exists a unique measurable vector function $w_\nu : X \to \overline{\mathbb{R}}^m$ with $|w_\nu(x)| = 1$ $|\nu|$ a.e. $x \in X$ such that $\nu = |\nu|_{w_\nu}$, that is

$$\nu(E) = \int_E w_\nu \, d|\nu| \quad \forall E \in \mathcal{M} \,.$$

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1.5. **Spaces $L^p(X,\mu)$ and their main properties. Riesz representation theorem.**

Definition of of $L^p(X,\mu)$ and L^p-norm.

Fisher-Riesz's theorem : $(L^p(X,\mu), \|\cdot\|_{L^p})$ is a B.s. if $1 \leqslant p \leqslant \infty$. Moreover $L^2(X,\mu)$ turns out to be a Hilbert space with respect to the scalar product

$$(f,g)_{L^2} := \int_X f\, g \, d\mu \quad f,\, g \in L^2(X,\mu) \,.$$

As a consequence of the proof of Riesz- Fisher's theorem we have the following useful result. Theorem : let $(f_h)_h \subset L^p(X,\mu)$ and $f \in L^p(X,\mu)$ with $1 \leqslant p \leqslant \infty$. Suppose that

$$\text{(MC)} \qquad \lim_{h\to\infty} \|f_h - f\|_{L^p(X,\mu)} = 0 \,.$$

Then, there exist a subsequence $(f_{h_k})_k$ and a function $g \in L^p(X,\mu)$ such that

(i) $f_{h_k}(x) \to f(x) \quad \mu -$ a.e. $x \in X$;
(ii) $|f_{h_k}(x)| \leqslant g(x) \quad \mu -$ a.e. $x \in X$, $\forall\, k$.

Theorem (Hölder inequality) ($\circ$): let p and p' be conjugate exponents, $1 \leqslant p < \infty$, that is

$$p' := \begin{cases} \dfrac{p}{p-1} & \text{if } 1 < p < \infty \\ \infty \quad \text{if } p = 1 \end{cases}$$

Let $f \in L^p(X,\mu)$ and $g \in L^{p'}(X,\mu)$. Then $f\,g \in L^1(X,\mu)$ and

$$\|fg\|_{L^1(X,\mu)} \leqslant \|f\|_{L^p(X,\mu)} \, \|g\|_{L^{p'}(X,\mu)}$$

Riesz representation theorem for the dual of L^p ($\circ$): if $1 < p < \infty$, then the mapping $T: L^{p'}(X,\mu) \to (L^p(X,\mu))'$, defined by

$$\langle T(u), f\rangle_{(L^p(X,\mu))' \times L^p(X,\mu)} := \int_X u\, f \, d\mu \quad \forall f \in L^p(X,\mu) \,,$$

is an isometric isomorphism, that is, T is a linear, one-to-one, onto mapping and

$$\|T(u)\|_{(L^p(X,\mu))'} = \|u\|_{L^{p'}(X,\mu)} \quad \forall u \in L^{p'}(X,\mu) \,.$$

If $p = 1$, the same conclusion holds under the additional assumption that μ is σ-finite. We will mean this feature by means of the identification

$$L^{p'}(X,\mu) \equiv (L^p(X,\mu))' \,. \tag{1.5}$$

Remark 1.1. Identification (1.5) may fail in the other cases.

Theorem (Approximation in L^p by continuous functions) : let $(X, \mathcal{B}(X), \mu)$ be a measure space with (X, d) l.c.s. and μ Radon measure. Then $C^0_c(X)$ is dense in $(L^p(X), \|\cdot\|_{L^p})$, provided that $1 \leqslant p < \infty$.

Definition of continuity for a linear functional $L : (\mathbf{C}^0_c(X))^m \to \mathbb{R}$ and its characterization by means of sequences.

Riesz representation theorem : let (X, d) be a separable, locally compact metric space and let $L : (\mathbf{C}^0_c(X))^m \to \mathbb{R}$ be a continuous linear functional. Then there exist a Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ and a Borel measurable vector function $w_L : X \to \mathbf{S}^{m-1}$ such that

$$L(u) = \int_X (w_L, u)_{\mathbb{R}^m} \, d\mu_L \quad \forall u \in (\mathbf{C}^0_c(X))^m \, , \tag{1.6}$$

that is, $L = w_L \mu$, and μ_L is characterized by the following identity: for each open set $A \subset X$

$$\mu_L(A) = \sup \{ L(u) : \, u \in (\mathbf{C}_c(X))^m, \, \mathrm{spt} u \subset A, \, \|u\|_\infty \leqslant 1 \} \, . \tag{1.7}$$

Moreover representation (1.6) is unique.

Definitions of Radon vector measure and finite Radon vector measure. Spaces of measures $(\mathcal{M}_{\mathrm{loc}}(X))^m$ and $(\mathcal{M}(X))^m$.

The Riesz representation theorem provides a characterization of the measures spaces $\mathcal{M}(X)^m$ and $(\mathcal{M}_{\mathrm{loc}}(X))^m$ as dual spaces of suitable spaces of continuous functions.

Corollary (Characterization of $\mathcal{M}_{\mathrm{loc}}(X)$): let (X, d) be a locally compact separable metric space and define

$$(C^0_c(X))' := \big\{ L : \, C^0_c(X) \to \mathbb{R} : \, L \text{ is linear and continuous} \big\} \, . \tag{1.8}$$

Let us define the map

$$I : \, \mathcal{M}_{\mathrm{loc}}(X) \to (C^0_c(X))' \quad I(\nu) := L_\nu \, . \tag{1.9}$$

Then I is an isomorphism (between vector spaces).

Proposition: Let $\nu \in \mathcal{M}(X)$, and let $L_\nu : \mathbf{C}^0_c(X) \to \mathbb{R}$ be the functional

$$L(u) := \int_X f \, d\nu \quad \forall \, u \in \mathbf{C}^0_c(X) \, .$$

. Then

$$\|L_\nu\|_{(\mathbf{C}^0_c(X), \|\cdot\|_\infty)'} := \sup \big\{ |L_\nu(u)| : \, u \in \mathbf{C}^0_c(X), \, \|u\|_\infty \leqslant 1 \big\} = |\nu|(X) \, .$$

Theorem (Characterization of $\mathcal{M}(X)$): let (X, d) be a locally compact separable metric space. Let I the map in (1.9). Then

(i) $I(\mathcal{M}(X)) = (C^0_0(X), \|\cdot\|_\infty)'$;

(iI) $I : \, \mathcal{M}(X) \to (C^0_0(X), \|\cdot\|_\infty)'$ is a topological isomorphism, that is an algebraic isomorphism, continuous with its inverse.

Corollary($\circ$): let (X, d) be a compact metric space. Then $(C^0(X), \|\cdot\|_\infty)'$ is isometrically isomporphic to $\mathcal{M}(X)$.

1.6. **Operations on measures.**

Definition of support for a positive or signed vector measure.

Definition of restriction of an outer measure, positive or vector signed measure to a set.

Theorem :

(i) If ν is an outer measure on X, then so is $\nu \llcorner E$. Moreover every ν-measurable set is also $\nu \llcorner E$-measurable.
(ii) If ν is a Borel regular outer measure on X and E is ν-measurable with $\nu(E) < \infty$, then $\nu \llcorner E$ is a Radon outer measure.
(iii) If ν is a positive or vector signed measure on a measure space $(X, \mathcal{M})$ and $E \in \mathcal{M}$, so is $\nu \llcorner E$.

Definition of push-forward for a positive or vector measure.

Example: Push-forward of the classical length measure.

Theorem (Cavalieri's principle): let $(X, \mathcal{M})$ be a measure space, μ a positive measure on it and $u : X \to [0, \infty]$ be measurable. Let $[0, \infty) \ni t \mapsto \mu(\{u > t\})$ denote the *distribution function of* u, that is,

$$\mu(\{u > t\} = \mu\left(\{x \in X : u(x) > t\}\right) \text{ if } t \in [0, \infty)\,. \tag{1.10}$$

Let $\theta : [0, \infty) \to [0, \infty)$ be (strictly) increasing such that $\theta(0) = 0$, $\theta : [0, T] \to [0, \infty)$ is absolutely continuous for each $T \in [0, \infty)$. Then

$$\int_X (\theta \circ u)\, d\mu = \int_0^\infty \theta'(t)\, \mu(\{u > t\})\, dt\,. \tag{1.11}$$

In particular, if $\theta(t) = t^p$ with $p \geqslant 1$, then

$$\int_X u^p\, d\mu = p \int_0^\infty t^{p-1}\, \mu(\{u > t\})\, dt\,.$$

1.7. **Weak*-convergence of measures. Regularization of Radon measures in $\mathbb{R}^n$.**

Definition of locally weakly* convergence of measures in $(\mathcal{M}_{\text{loc}}(X))^m$ and of weakly* convergence in $(\mathcal{M}(X))^m$.

Proposition (Locally weak* convergence vs. weak* convergence): assume that $(\nu_h)_h$, $\nu \subset \mathcal{M}_{\text{loc}}(X)$. Then they are equivalent:

(i) $\nu_h \overset{*}{\rightharpoonup} \nu$ and $\sup_h |\nu_h|(X) < \infty$;
(ii) $(\nu_h)_h$, $\nu \subset \mathcal{M}(X)$ and $(\nu_h)_h$ weakly* converges to ν.

Other interesting examples of weak*-converging sequences of measures illustrating a wide variety of behaviours can be found in [Mag, Examples 4.20-4.23].

Theorem (Characterziation of the locally weak* convergence of positive Radon measures): let $(\mu_h)_h$ and μ be positive Radon measures on $(X, \mathcal{B}(X))$.Then the following are equivalent.

(i) $\mu_h \overset{*}{\rightharpoonup} \mu$ as $h \to \infty$.

(ii) If K compact and A open, then

$$\mu(K) \geqslant \limsup_{h\to\infty} \mu_h(K)\,, \tag{1.12}$$

$$\mu(A) \leqslant \liminf_{h\to\infty} \mu_h(A)\,. \tag{1.13}$$

(iii) If $E \in \mathcal{B}_{\rm comp}(X)$ with $\mu(\partial E) = 0$, then

$$\mu(E) = \lim_{h\to\infty} \mu_h(E)\,.$$

Moreover, if $\mu_h \overset{*}{\rightharpoonup} \mu$ as $\to \infty$, then for every $x \in {\rm spt}\mu$ there exists $(x_h)_h \subset X$ with

$$\lim_{h\to\infty} x_h = x, \quad x_h \in {\rm spt}\mu_h \quad \forall\, h \in \mathbb{N}\,. \tag{1.14}$$

Proposition (Characterization of the narrow convergence of positive Radon measures) : let $(\mu_h)_h$ be a sequence of positive, finite Radon measures on $(X, \mathcal{B}(X))$ and assume the existence of a positive, finite Radon measure μ such that

$$\lim_{h\to\infty} \mu_h(X) = \mu(X) \text{ and } \liminf_{h\to\infty} \mu_h(A) \geqslant \mu(A) \tag{1.15}$$

for every $A \subset X$ open set. Then

$$\lim_{h\to\infty} \int_X u\, d\mu_h = \int_X u\, d\mu \quad \forall\, u \in \mathbf{C}^0_b(X) \tag{NC}$$

where $\mathbf{C}^0_b(X)$ denotes the class of all bounded continuous function $u : X \to \mathbb{R}$. In particular $(\mu_h)_h$ weakly* converges to μ. Moreover if (NC) holds so does (1.15), that is (NC) and (1.15) are equivalent.

Theorem : let $(\nu_h)_h$ and ν be $\mathbb{R}^m$-valued Radon vector measures, that is $\nu : \mathcal{B}(X) \to \mathbb{R}^m$, and let μ a positive Radon measure on a l.c.s. metric space (X, d).

(i) If $\nu_h \overset{*}{\rightharpoonup} \nu$, then for every open set $A \subset X$

$$|\nu|(A) \leqslant \liminf_{h\to\infty} |\nu_h|(A)\,. \tag{1.16}$$

(ii) If $\nu_h \overset{*}{\rightharpoonup} \nu$ and $|\nu_h| \overset{*}{\rightharpoonup} \mu$, then

$$|\nu|(B) \leqslant \mu(B) \quad \forall\, E \in \mathcal{B}(X)\,. \tag{1.17}$$

Moreover, if $E \in \mathcal{B}_{\rm comp}(X)$ with $\mu(\partial E) = 0$, then

$$\nu(E) = \lim_{h\to\infty} \nu_h(E)\,.$$

(iii) If $\nu_h \overset{*}{\rightharpoonup} \nu$ and $\lim_{h\to\infty} |\nu_h|(X) = |\nu|(X) < \infty$, then (NC) holds with $\mu_h = |\nu_h|$ and $\mu = |\nu|$. In particular $|\nu_h| \overset{*}{\rightharpoonup} |\nu|$.

Theorem (Weak*-compactness): if $(\nu_h)_h$ is a sequence of $\mathbb{R}^m$- valued finite Radon measures on the l.c.s. metric space X, that is $(\nu_h)_h \subset (\mathcal{M}(X))^m$, with $\sup_h |\nu_h|(X) < \infty$, then it has a weakly*- converging subsequence. Moreover, the map $\nu \mapsto |\nu|$ is lower semicontinuous with respect the weak*-convergence.

Corollary (Local weak* compactness): let $(\nu_h)_h$ be a sequence of $\mathbb{R}^m$- valued Radon measures on the l.c.s. metric space X, $(\nu_h)_h \subset (\mathcal{M}_{loc}(X))^m$, such that

$$\sup\{|\nu_h|(K) : h \in \mathbb{N}\} < \infty$$

for every compact $K \subset X$; then it has a locally weakly*-converging subsequence.

2. Differentiation of Radon measures

2.1. Covering theorems and Vitali-type covering property for measures on $\mathbb{R}^n$.

Definition of a cover and of a fine cover in a metric space.

Theorem (Vitali covering theorem): let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$ with

$$D = \sup\{d(B) : \, B \in \mathcal{G}\} < \infty \,.$$

Then there exists a (pairwise) disjoint family $\mathcal{F} \subseteq \mathcal{G}$, which is at most countable, such that

$$\cup_{B\in\mathcal{G}} B \subset \cup_{B\in\mathcal{F}} \hat{B} \,.$$

where $\hat{B}$ is an enlargement of B, that is $\hat{B} = 5B$.

Theorem (Vitali covering property for the Lebesgue measure): Let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$, which is a fine cover of a (possibly non measurable) set $A \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Then there exists a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that

$$\mathcal{L}^n \left(A \setminus \cup\mathcal{F}\right) = 0 \,,$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue outer measure.

Example: Vitali covering property does not hold for all Radon measures in $\mathbb{R}^n$

Theorem (Besicovitch's covering theorem): there are integers $P(n)$and $Q(n)$ depending only on n with the following properties. Let A be a bounded subset of $\mathbb{R}^n$, and let $\mathcal{G}$ be a family of closed balls such that each point of A is the centre of some ball of $\mathcal{G}$.

(i) There is a finite or countable subfamily $\mathcal{F} \subset \mathcal{G}$ which covers A and every point of $\mathbb{R}^n$ belongs to at most $P(n)$ balls of $\mathcal{F}$, that is,

$$\chi_A \leqslant \sum_{B\in\mathcal{F}} \chi_B \leqslant P(n) \,.$$

(ii) There are subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{Q(n)} \subset \mathcal{G}$ covering A such that each $\mathcal{F}_i$ is disjoint, that is,

$$A \subset \cup_{i=1}^{Q(n)} \left(\cup\mathcal{F}_i\right)$$

and

$$B \cap B' = \emptyset \text{ for } B, B' \in \mathcal{F}_i \text{ with } B \neq B' \,.$$

Theorem (Vitali covering property for Radon measures): let φ be a Radon o. m. in $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ (even not φ-measurable) and $\mathcal{G}$ a family of closed balls. Assume that $\mathcal{G}$ is cover of A and

$$\inf\{\, r : \, B(x,r) \in \mathcal{G}\,\} = 0 \quad \forall\, x \in A \,. \tag{2.1}$$

Then there is a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that

$$\varphi\left(A \setminus \cup\mathcal{F}\right) = 0 \,.$$

2.2. **Derivatives of Radon measures on $\mathbb{R}^n$. Lebesgue-Besicovitch differentiation theorem for Radon measures on $\mathbb{R}^n$.**

Definition of upper/lower derivatives and of derivative of a positive Radon measure with respect to an other one.

Theorem (Differentation for positive Radon measures): let ν and μ be positive Radon measures on $\mathbb{R}^n$.

(i) The derivative $D_\mu\nu(x)$ exists and is finite (that is $D_\mu\nu(x) \in [0,\infty)$) for μ-a.e. $x \in \mathbb{R}^n$.

(ii) The function $D_\mu\nu : \mathbb{R}^n \to [0,+\infty]$ is Borel measurable, by defining $D_\mu\nu = \infty$ on the possible μ-negligible set where it does not exist.

(iii) Let

$$A := \{x \in \mathbb{R}^n : \exists\, D_\mu\nu(x) \in [0,\infty)\} \,. \tag{2.2}$$

For all Borel sets $B \subset \mathbb{R}^n$

$$\int_B D_\mu\nu\, d\mu = \nu(A \cap B) \leqslant \nu(B)\,, \tag{2.3}$$

with equality if $\nu << \mu$. In this case

$$D_\mu\nu(x) = \frac{d\nu}{d\mu}(x) = \frac{d\nu_{ac}}{d\mu}(x) \quad \mu\text{-a.e. } x \in \mathbb{R}^n\,.$$

denoting $\dfrac{d\lambda}{d\mu}$ the Radon-Nikodym derivative of λ with respect to μ.

(iv) $\nu << \mu$ if and only if $\underline{D}_\mu\nu(x) < \infty$ ν-a.e. $x \in \mathbb{R}^n$.

Theorem (Lebesgue-Besicovitch differentiation theorem): let μ be a positive Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\exists \lim_{r\to 0} \fint_{B(x,r)} f(y)\, d\mu(y) = f(x) \quad \mu-\text{a.e. } x \in \mathbb{R}^n\,,$$

that is, by definition, there exists a μ-negligible set $N \subset \mathbb{R}^n$ (i.e. $\mu(N) = 0$) such that

$$\exists \lim_{r\to 0} \fint_{B(x,r)} f(y)\, d\mu(y) = f(x) \quad \forall\, x \in \mathbb{R}^n \setminus N\,. \tag{$\star$}$$

Theorem (of Lebesgue points): let μ be a positive Radon measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and let $f \in L^1_{\rm loc}(\mathbb{R}^n, \mu)$. Then μ-a.e. $x \in \mathrm{spt}(\mu)$ there exists

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\mu(y) = 0\,. \tag{LP}$$

Corollary (Density of a set): let μ be a positive Radon measure on $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ be a measurable set. Then

$$\exists \lim_{r\to 0} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} = \begin{cases} 1 & \text{for } \mu\text{-a.e. } x \in E \\ 0 & \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n \setminus E \end{cases},$$

that is, μ-a.e. $x \in E$ is a point of density 1 for E and μ-a.e. $x \in \mathbb{R}^n \setminus E$ is a point of density 0 for E.

Theorem (Lebesgue decomposition in terms of derivatives of measures): let ν and μ be positive Radon measures on $\mathbb{R}^n$. Let ν_{ac} and ν_s denote respecitively the absolutely continuous and singular parts of ν in the Lebesgue decomposition with respect to μ. Then, for each Borel set B,

$$\nu_{ac}(B) = \int_B D_\mu \nu(x)\, d\mu, \quad \nu_s(B) = \nu \llcorner S(B) = \nu(S \cap B)$$

where S is the μ-negligible Borel set

$$S = (\mathbb{R}^n \setminus \mathrm{spt}(\mu)) \cup \{x \in \mathrm{spt}(\mu) :\, D_\mu \nu(x) = \infty\}$$

Definition of a regualr differenziation basis for a positive Radon measure on $\mathbb{R}^n$ $(\star)$.

Theorem (Differentiation for positive Radon measures with respect to a differentiation basis): let ν and μ be postive Radon measures on $\mathbb{R}^n$, then there exists

$$\lim_{h\to\infty} \frac{\nu(E_h(x))}{\mu(E_h(x))} = \frac{d\nu_{ac}}{d\mu}(x) = D_\mu \nu_{ac}(x) \quad \mu\text{-a.e. } x \in \mathrm{spt}(\mu)\,,$$

whenever $(E_h(x))_h$ is a differentiation basis of μ at x.

Theorem: let ν and μ be respectively a $\mathbb{R}^m$-valued Radon and a positive Radon measures on $\mathbb{R}^n$. Then, for μ-a.e. $x \in \mathrm{spt}(\mu)$,

$$\exists\, w(x) := \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \in \mathbb{R}^m\,. \tag{2.4}$$

Moreover the Lebsesgue decomposition of ν with respect to μ is given by $\nu = \mu_w + \nu_s$ where $\nu_s(B) = \nu \llcorner S(B) = \nu(S \cap B)$ and S is the μ-negligible Borel set

$$S = (\mathbb{R}^n \setminus \mathrm{spt}(\mu)) \cup \{x \in \mathrm{spt}(\mu) :\, D_\mu |\nu|(x) = \infty\}$$

3. An introduction to Hausdorff measures.

3.1. Carathéodory's construction and definition of Hausdorff measures on a metric space and their elementary properties; Hausdorff dimension.

Caratheodory's construction in a metric space (X,d) and set function $\psi = \psi(\mathcal{F},\zeta) : \mathcal{P}(X) \to [0,\infty]$.

Theorem

(i) ψ is a Borel outer measure.
(ii) If $\mathcal{F} \subset \mathcal{B}(X)$, then ψ is a Borel regular outer measure.

Definition of s-dimensional Hausdorff pre-measure $\mathcal{H}^s_\delta : \mathcal{P}(X) \to [0,\infty]$, measure $\mathcal{H}^s : \mathcal{P}(X) \to [0,\infty]$ and spherical Hausdorff measure $\mathcal{S}^s : \mathcal{P}(X) \to [0,\infty]$ in a separable metric space (X,d).

Comparison between $\mathcal{H}^s$ and $\mathcal{S}^s$: $\mathcal{H}^s(A) \leqslant \mathcal{S}^s(A) \leqslant 2^s\, \mathcal{H}^s(A)\ \forall\, A \subset X$.

Theorem : let $s \in [0,\infty)$, $\alpha_s > 0$ and $\zeta(E) := \alpha_s\, d(E)^s$ for $E \subset X$. If

(i) $\mathcal{F} = \{F \subset X : F \text{ closed }\}$
or
(ii) $\mathcal{F} = \{U \subset X : F \text{ open }\}$,
then $\psi(\mathcal{F}, \zeta) = \mathcal{H}^s$.

Corollary($\circ$): $\mathcal{H}^s$ is a Borel regular outer measure.

Lemma ($\mathcal{H}^s$-null sets): Let $A \subset X$, $0 < s < \infty$ and $0 < \delta \leqslant \infty$. Then the following conditions are equivalent:
(i) $\mathcal{H}^s(A) = 0$.
(ii) $\mathcal{H}^s_\delta(A) = 0$.
(iii) $\forall \epsilon > 0 \, \exists E_1, E_2, \cdots \subset X$ such that

$$A \subset \cup_{i=1}^{\infty} E_i \text{ and } \sum_{i=1}^{\infty} d(E_i)^s < \epsilon \, .$$

Theorem : for $0 < s < t < \infty$ and $A \subset X$,
(i) $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$,
(ii) $\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = \infty$.

Definition of Hausdorff metric in a separable metric space and its basic properties.

3.2. **Recalls of some fundamental results on Lipschitz functions between Euclidean spaces and relationships with Hausdorff measures.**

Definition of Lipschitz function bewteen metric spaces.

Theorem (Whitney's extension theorem): let C be a closed set in $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ and $v : C \to \mathbb{R}^n$ be continuous functions. Define

$$R(x,y) := \frac{f(x) - f(y) - v(y) \cdot (x-y)}{|x-y|} \quad \forall \, x, y \in C, x \neq y \, .$$

Suppose that for all compact sets $K \subset C$

$$\lim_{r \to 0} \sup \{ |R(x,y)| : x, y \in K, 0 < |x-y| < r \} = 0 \, . \tag{3.1}$$

Then there is $\hat{f} \in \mathbf{C}^1(\mathbb{R}^n)$ such that $\hat{f}|_C = f$ and $\nabla \hat{f}|_C = v$.

Theorem (Mc Shane's extension theorem): let (X, d) be a metric space and $f : E \subset X \to \mathbb{R}$ be L-Lipschitz. Then there is $\hat{f} : X \to \mathbb{R}$ such that $\hat{f}|_E = f$ and $\hat{f}$ is L-Lipschitz.

Corollary: let $f : E \to \mathbb{R}^m$, $E \subset (X, d)$ be an L-Lipschitz function. Then there exists an $\sqrt{m}L$-Lipschitz function $\hat{f} : X \to \mathbb{R}^m$ such that $\hat{f}|_E = f$.

Theorem (Kirszbraun's theorem) ($\circ$): let $f : E \to \mathbb{R}^m$, $E \subset \mathbb{R}^n$, be an L-Lipschitz function. Then there exists an L-Lipschitz function $\hat{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\hat{f}|_E = f$.

Theorem (Rademacher's theorem[Rad]): let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then f is differentiable (in classical sense) $\mathcal{L}^n$-a.e., that is,

$$\exists\, \nabla f(x) := (\partial_1 f(x), \ldots, \partial_n f(x)) \ \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n$$

and

$$\lim_{y \to x} \frac{f(y) - f(x) - df(x)(y-x)}{|y-x|} = 0 \tag{3.2}$$

where $df(x) : \mathbb{R}^n \to \mathbb{R}$ denotes the (linear) differential map of f at x defined by

$$df(x)(v) := \nabla f(x) \cdot v \quad \forall\, v \in \mathbb{R}^n\,.$$

Moreover $\nabla f \in (L^\infty_{\mathrm{loc}}(\mathbb{R}^n))^n$.

Theorem (Approximation of Lipschitz functions) :Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then for each $\epsilon > 0$ there is a $g \in \mathbf{C}^1(\mathbb{R}^n)$ such that

$$\mathcal{L}^n\left(\{x : f(x) \neq g(x)\} \cup \{x : \nabla f(x) \neq \nabla g(x)\}\right) < \epsilon\,.$$

In addition, there is a positive constant $c = c(n)$ such that

$$\sup_{\mathbb{R}^n} |\nabla g| \leqslant c\, Lip(f)\,.$$

Theorem (Hausdorff measures vs. Lipschitz maps): let $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. Then

$$\mathcal{H}^s(f(E)) \leqslant Lip(f)^s\, \mathcal{H}^s(E) \quad \forall\, 0 \leqslant s < \infty\,.$$

In particular

$$\mathrm{Hdim}(f(E)) \leqslant \mathrm{Hdim}(E)$$

3.3. Hausdorff measures in the Euclidean spaces; $\mathcal{H}^1$ and the classical notion of length in $\mathbb{R}^n$; isodiametric inequality and identity $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.

Definition of s-dimensional Hausdorff measure in $\mathbb{R}^n$.

s-dimensional Hausdorff measures are invariant by translations and s-honogeneous by dilations in $\mathbb{R}^n$.

Theorem (Classical length and $\mathcal{H}^1$): let $\gamma : [0, a] \to \mathbb{R}^n$ be a curve and denote $\Gamma = \gamma([0, a])$ its support. Then

$$\mathcal{H}^1(\Gamma) \leqslant l(\gamma)$$

and equality holds if γ is injective.

Proposition:

$$\mathcal{H}^s(B(x, r)) = c(s, n)\, r^s \quad x \in \mathbb{R}^n,\ 0 < r < \infty \tag{3.3}$$

with $c(s, n)$ positive and finite constant only when $s = n$; for $s > n$, $c(s, n) = 0$; for $s < n$, $c(s, n) = \infty$.

Corollary :

(i) $\mathcal{H}^s$ is a (non trivial) Radon measure on $\mathbb{R}^n$ if and only $s = n$.

(ii) $\mathrm{Hdim}(A) = n$ for each (nonemtpy) open set $A \subset \mathbb{R}^n$. In particular $\mathrm{Hdim}(\mathbb{R}^n) = n$.

Theorem (Isodiametric inequality):

$$\mathcal{L}^n(A) \leqslant \alpha_n\, d(A)^n \quad \text{for } A \subset \mathbb{R}^n\,.$$

Theorem ($\mathcal{H}^n \equiv \mathcal{L}^n$): $\mathcal{L}^n(A) = \mathcal{H}^n_\delta(A) = \mathcal{H}^n(A)$ for each $A \subset \mathbb{R}^n$, $0 < \delta \leqslant \infty$.

Definition of Cantor sets $C(\lambda)$ in $\mathbb{R}$ if $\lambda \in (0, 1/2)$: their basic properties and self-similar structure.

Theorem (Hausdorff dimension of the Cantor sets in $\mathbb{R}$): let

$$s = \frac{\log 2}{\log \frac{1}{\lambda}}$$

and let α_s be the constant in the definition of s-dimensional Hausdorff measure. Then

(i) $\mathcal{H}^s\left(C(\lambda)\right) \leqslant \alpha_s < \infty$;

(ii) $\mathcal{H}^s\left(C(\lambda)\right) \geqslant \alpha_s > 0$,

In particular $\mathcal{H}^s\left(C(\lambda)\right) = \alpha_s$ and

$$\mathrm{Hdim}(C(\lambda)) = \frac{\log 2}{\log \frac{1}{\lambda}}\,.$$

Definition of k-dimensional upper/lower density, denoted, respectively, $\Theta^*_k(\mu, \cdot)$ and $\Theta_{*k}(\mu, \cdot)$, and density, denoted $\Theta_k(\mu, \cdot)$, for a Radon measure μ on an open set of $\mathbb{R}^n$. If $\mu = \mathcal{H}^k \llcorner E$, we set

$$\Theta^*_k(E, \cdot) := \Theta^*_k(\mu, \cdot),\ \Theta_{*k}(E, \cdot) := \Theta_{*k}(\mu, \cdot),\ \Theta_k(E, \cdot) := \Theta_k(\mu, \cdot)\,.$$

Theorem (Estimates of the upper density of a Radon measure): let $\Omega \subset \mathbb{R}^n$ be an open set and μ a positive Radon measure in Ω. Then, for any $t \in (0, \infty)$ and any Borel set $B \subset \Omega$ the following implications hold:

$$\Theta^*_k(\mu, x) \geqslant t \quad \forall\, t \in B \quad \Rightarrow \quad \mu \geqslant t\, \mathcal{H}^k \llcorner B\,, \tag{3.4}$$

$$\Theta^*_k(\mu, x) \leqslant t \quad \forall\, t \in B \quad \Rightarrow \quad \mu \leqslant 2^k\, t\, \mathcal{H}^k \llcorner B\,. \tag{3.5}$$

Corollary: let $k \in [0, n]$ and assume that $E \subset \mathbb{R}^n$ is $\mathcal{H}^k$-measurable and $\mathcal{H}^k(E) < \infty$. Then

$$\exists\, \Theta_k(E, x) = \lim_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B(x, r))}{\alpha^*_k\, r^k} = 0 \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in \mathbb{R}^n \setminus E\,; \tag{3.6}$$

$$2^{-k} \leqslant \limsup_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B(x, r))}{\alpha^*_k\, r^k} \leqslant 1 \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in E\,. \tag{3.7}$$

3.4. **Area and coarea formulas in $\mathbb{R}^n$ and some applications.**

Definition of Jacobian function $\mathbf{J}f : \mathbb{R}^n \to [0,\infty]$ for a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$.

Theorem (Area formula): let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz with $n \leqslant m$. Then for each $\mathcal{L}^n$-measurable subset $A \subset \mathbb{R}^n$

$$\text{(AF)} \qquad \int_A \mathbf{J}f \, d\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^0\left(A \cap f^{-1}(y)\right) \, d\mathcal{H}^n(y)\,.$$

Theorem (Area formula for injective maps): let $n \leqslant m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an injective Lipschitz function and $A \subset \mathbb{R}^n$ be a measurable set. Then

$$\text{(IAF)} \qquad \mathcal{H}^n(f(A)) = \int_A \mathbf{J}f \, d\mathcal{L}^n$$

and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$.

Some applications of the area formula: lenght of a curve and area of a graph.

Theorem (Change of variables): let $n \leqslant m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. The for each $\mathcal{L}^n$-integrable function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] d\mathcal{H}^n(y)\,.$$

In particular, if f is injective,

$$\int_{\mathbb{R}^n} g(x)\, \mathbf{J}f(x)\, d\mathcal{L}^n(x) = \int_{f(\mathbb{R}^n)} g(f^{-1}(y))\, d\mathcal{H}^n(y)\,.$$

Theorem (Corea formula): let $n \geqslant m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. The for each $\mathcal{L}^n$-measurable set $A \subset \mathbb{R}^n$

$$\int_A \mathbf{J}f(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(y))\, d\mathcal{L}^m(y)\,.$$

Remark: applying the coarea formula to set $A := \{x \in \mathbb{R}^n : \mathbf{J}f(x) = 0\}$, we get that

$$\text{(WMS)} \qquad \mathcal{H}^{n-m}\left(\{\mathbf{J}f = 0\} \cap f^{-1}(y)\right) = 0 \quad \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m\,.$$

This is a weak variant of Morse-Sard's theorem which asserts

$$\text{(MS)} \qquad \{\mathbf{J}f = 0\} \cap f^{-1}(y) = \emptyset \quad \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m\,,$$

provided that $f \in \mathbf{C}^k(\mathbb{R}^n; \mathbb{R}^m)$ for $k = 1 + n - m$.

Theorem (Change of variables formula): under the same assumptions of the Coarea Formula. Then for each $\mathcal{L}^n$-measurable function $g : \mathbb{R}^n \to \mathbb{R}$

(i) $g|_{f^{-1}(y)}$ is $\mathcal{H}^{n-m}$-summable $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$.

(ii)

$$\int_{\mathbb{R}^n} g(x)\,\mathbf{J}f(x)\,d\mathcal{L}^n(x) = \int_{\mathbb{R}^m}\left[\int_{f^{-1}(y)} g(x)\,d\mathcal{H}^{n-m}(x)\right]\,d\mathcal{L}^m(y)\,,$$

4. Rectifiable sets and blow-ups of Radon measures

Definition of k-dimensional planes and the orthogonal group in $\mathbb{R}^n$.

Definition of countably $\mathcal{H}^k$-rectifiable, locally $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k$-rectifiable set in $\mathbb{R}^n$.

Example of a rectifibale set: a Lipschitz k-graph.

4.1. **Rectifiable sets of $\mathbb{R}^n$ and their decomposition in Lipschitz images.**

Definition of regular Lipschitz image.

Theorem (Decomposition of rectifiable sets): if Γ is countably $\mathcal{H}^k$- rectifiable in $\mathbb{R}^n$ and $t > 1$, then there exist a Borel set $\Gamma_0 \subset \mathbb{R}^n$, countably many Lipschitz maps $f_h : \mathbb{R}^k \to \mathbb{R}^n$ and compact sets $E_h \subset \mathbb{R}^k$ such that

$$\Gamma = \Gamma_0 \cup (\cup_{h=1}^{\infty} f(E_h))\,, \quad \mathcal{H}^k(\Gamma_0) = 0\,.$$

Each pair (f_h, E_h) defines a regular Lipschitz image, with $Lip(f_h) \leqslant t$ and

$$t^{-1}|x-y| \leqslant |f_h(x) - f_h(y)| \leqslant t\,|x-y|,$$
$$t^{-1}\,|v| \leqslant |Df_h(x)v| \leqslant t\,|v|,$$
$$t^{-k} \leqslant \mathbf{J}f_h(x) \leqslant t^k,$$

for every x, $y \in E_h$ and $v \in \mathbb{R}^k$.

4.2. **Approximate tangent planes to rectifiable sets.**

Theorem (Existence of approximate tangent spaces): let

$$\Phi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n,\ \Phi_{x,r}(y) := \frac{y-x}{r},\ y \in \mathbb{R}^n\,.$$

If $\Gamma \subset \mathbb{R}^n$ is a locally $\mathcal{H}^k$-rectifiable set, then for $\mathcal{H}^k$-a.e. $x \in \Gamma$ there exists a unique k- dimensional plane π_x such that, as $r \to 0^+$,

$$\frac{(\Phi_{x,r})_{\#}\mu}{r^k} = \mathcal{H}^k \llcorner \left(\frac{\Gamma - x}{r}\right) \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x\,,$$

that is

$$\lim_{r\to 0^+} \frac{1}{r^k}\int_{\Gamma} \varphi\left(\frac{y-x}{r}\right)\,d\mathcal{H}^k(y) = \int_{\pi_x} \varphi(y)\,d\mathcal{H}^k(y) \quad \forall\,\varphi \in \mathbf{C}^0_c(\mathbb{R}^n)\,.$$

In particular

$$\exists\,\Theta_k(\Gamma, x) := \lim_{r\to 0^+} \frac{\mathcal{H}^k(\Gamma \cap B(x,r))}{\alpha^*_k\, r^k} = 1 \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in \Gamma\,.$$

Definition of the approximate tangent plane to a subset at a point

Proposition (Locality of the approximate tangent plane):If Γ_i $(i=1,2)$ are locally $\mathcal{H}^k$-rectifiable sets of $\mathbb{R}^n$, then for $\mathcal{H}^k$-a.e. $x\in\Gamma_1\cap\Gamma_2$

$$T_x\Gamma_1 = T_x\Gamma_2\,.$$

Theorem (Besicovitch-Marstrand-Mattila): let E a Borel set with $\mathcal{H}^k(E) < \infty$. Then the following are equivalent:

(i) E is $\mathcal{H}^k$-rectifiable;

(ii) there exists $\Theta_k(E,x) = 1$ for $\mathcal{H}^k$-a.e. $x\in E$.

4.3. **Blow-ups of Radon measures on $\mathbb{R}^n$ and rectifiability.**

Definition of cone $K(\pi,t)$ in $\mathbb{R}^n$.

Theorem (Rectifiability criterion): if $\Gamma\subset\mathbb{R}^n$ is a compact set, π is a k-dimensional plane in $\mathbb{R}^n$, and there exist δ and t positive with

$$\Gamma\cap B(x,\delta)\subset x+K(\pi,t)\quad\forall\, x\in\Gamma\,, \tag{4.1}$$

then Γ is $\mathcal{H}^k$-rectifiable, since there exist finitely many Lipschitz maps $f_h:\mathbb{R}^k\to\mathbb{R}^n$ $(h=1,\dots,N)$ and compact sets $F_h\subset\mathbb{R}^k$ with

$$\Gamma=\cup_{h=1}^N f_h(F_h)\,.$$

Theorem (Rectifiability by convergence of the blow-ups): If μ is a Radon measure on $\mathbb{R}^n$, Γ is a Borel set in $\mathbb{R}^n$, μ is concentrated on Γ (that is $\mu=\mu\llcorner\Gamma$), and, for every $x\in\Gamma$, there exists a k-dimensional plane π_x in $\mathbb{R}^n$ such that

$$\frac{(\Phi_{x,r})_\#\mu}{r^k}\overset{*}{\rightharpoonup}\mathcal{H}^k\llcorner\pi_x \text{ as } r\to 0^+,$$

then $\mu=\mathcal{H}^k\llcorner\Gamma$ and Γ is locally $\mathcal{H}^k$-rectifiable.

Definition of purely unrectifiable set,

Example of an unrectifiable set.

Results with proofs that will be a part of the interview: the student has to agree with the teacher two results, which do not belong to the same chapter of the programme.

- 1. Riesz representation theorem: let (X,d) be a separable, locally compact metric space and let $L : (\mathbf{C}_c^0(X))^m \to \mathbb{R}$ be a continuous linear functional. Then there exist a Radon measure $\mu : \mathcal{B}(X) \to [0,\infty]$ and a Borel measurable vector function $w_L : X \to \mathbf{S}^{m-1}$ such that

$$(\star) \qquad L(u) = \int_X (w_L, u)_{\mathbb{R}^m}\, d\mu_L \quad \forall u \in (\mathbf{C}_c^0(X))^m\,,$$

  that is, $L = w_L\mu$, and μ_L is characterized by the following identity: for each open set $A \subset X$

$$\mu_L(A) = \sup\{L(u) : u \in (\mathbf{C}_c(X))^m,\ \mathrm{spt} u \subset A,\ \|u\|_\infty \leqslant 1\}\ .$$

  Moreover representation is unique.
- 2. Theorem (Characterization of the locally weak* convergence of positive Radon measures): let $(\mu_h)_h$ and μ be positive Radon measures on $(X, \mathcal{B}(X))$.Then the following are equivalent.
  (i) $\mu_h \overset{*}{\rightharpoonup} \mu$ as $h \to \infty$.
  (ii) If K compact and A open, then

$$\mu(K) \geqslant \limsup_{h\to\infty} \mu_h(K)\,,$$

$$\mu(A) \leqslant \liminf_{h\to\infty} \mu_h(A)\,.$$

  (iii) If $E \in \mathcal{B}_{\mathrm{comp}}(X)$ with $\mu(\partial E) = 0$, then

$$\mu(E) = \lim_{h\to\infty} \mu_h(E)\,.$$

  Moreover, if $\mu_h \overset{*}{\rightharpoonup} \mu$ as $\to \infty$, then for every $x \in \mathrm{spt}\mu$ there exists $(x_h)_h \subset X$ with

$$\lim_{h\to\infty} x_h = x, \quad x_h \in \mathrm{spt}\mu_h \quad \forall\, h \in \mathbb{N}\,.$$

- 3. Theorem (Weak*-compactness): if $(\nu_h)_h$ is a sequence of $\mathbb{R}^m$- valued finite Radon measures on the l.c.s. metric space X, that is $(\nu_h)_h \subset (\mathcal{M}(X))^m$, with $\sup_h |\nu_h|(X) < \infty$, then it has a weakly*- converging subsequence. Moreover, the map $\nu \mapsto |\nu|$ is lower semicontinuous with respect the weak*-convergence.
- 4. Theorem (Vitali covering theorem) Let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$ with

$$D = \sup\{d(B) : B \in \mathcal{G}\} < \infty\,.$$

  Then there exists a (pairwise) disjoint family $\mathcal{F} \subseteq \mathcal{G}$, which is at most countable, such that

$$\cup_{B\in\mathcal{G}} B \subset \cup_{B\in\mathcal{F}} \hat{B}\,.$$

  where $\hat{B}$ is an enlargement of B, that is $\hat{B} = 5B$.

- 5. Theorem (Vitali covering property for the Lebesgue measure): Let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$, which is a fine cover of a (possibly non measurable) set $A \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Then there exists a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that
$$\mathcal{L}^n \left(A \setminus \cup \mathcal{F}\right) = 0\,,$$
where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue outer measure.
- 6. Theorem (Vitali covering property for Radon measures): let φ be a Radon o. m. in $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ (even not φ-measurable) and $\mathcal{G}$ a family of closed balls. Assume that $\mathcal{G}$ is cover of A and
$$\inf\{\, r : \, B(x,r) \in \mathcal{G}\,\} = \, 0 \quad \forall\, x \in A\,.$$
Then there is a disjoint subfamily $\mathcal{F} \subset \mathcal{G}$, at most countable, such that
$$\varphi\,(A \setminus \cup \mathcal{F}) = \, 0\,.$$
- 7. Theorem (Differentation for positive Radon measures):let ν and μ be positive Radon measures on $\mathbb{R}^n$.
  (i) The derivative $D_\mu \nu(x)$ exists and is finite (that is $D_\mu\nu(x) \in [0,\infty)$) for μ-a.e. $x \in \mathbb{R}^n$.
  (ii) The function $D_\mu\nu \, : \, \mathbb{R}^n \, \to \, [0,+\infty]$ is Borel measurable, by defining $D_\mu\nu = \, \infty$ on the possible μ-negligible set where it does not exist.
  (iii) Let
$$A := \, \{x \in \mathbb{R}^n : \, \exists\, D_\mu\nu(x) \in [0,\infty)\}\ .$$
  For all Borel sets $B \subset \mathbb{R}^n$
$$\int_B D_\mu \nu \, d\mu = \, \nu(A \cap B) \leqslant \, \nu(B)\,,$$
  with equality if $\nu << \mu$. In this case
$$D_\mu\nu(x) = \, \frac{d\nu}{d\mu}(x) = \, \frac{d\nu_{ac}}{d\mu}(x) \quad \mu\text{-a.e. } x \in \mathbb{R}^n\,.$$
  denoting $\dfrac{d\lambda}{d\mu}$ the Radon-Nikodym derivative of λ with respect to μ.
  (iv) $\nu << \mu$ if and only if $\underline{D}_\mu\nu(x) < \, \infty$ ν-a.e. $x \in \mathbb{R}^n$.
- 8. Theorem (Rademacher's theorem[Rad]): let $f \, : \, \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then f is differentiable (in classical sense) $\mathcal{L}^n$-a.e., that is,
$$\exists\, \nabla f(x) := (\partial_1 f(x), \ldots, \partial_n f(x)) \text{ for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n$$
and
$$\lim_{y \to x} \frac{f(y) - f(x) - df(x)(y-x)}{|y-x|} = \, 0\,,$$
where $df(x) : \, \mathbb{R}^n \to \mathbb{R}$ denotes the (linear) differential map of f at x defined by
$$df(x)(v) := \nabla f(x) \cdot v \quad \forall\, v \in \mathbb{R}^n\,.$$
Moreover $\nabla f \in (L^\infty_{\mathrm{loc}}(\mathbb{R}^n))^n$.
  (The suggested proof is in [EG, Theorem 2, Sect. 3.1.2].)

- 9. Theorem (Classical length and $\mathcal{H}^1$): let $\gamma : [0,a] \to \mathbb{R}^n$ be a curve and denote $\Gamma = \gamma([0,a])$ its support. Then
$$\mathcal{H}^1(\Gamma) \leqslant l(\gamma)$$
and equality holds if γ is injective.
- 10. Theorem ($\mathcal{H}^n \equiv \mathcal{L}^n$): $\mathcal{L}^n(A) = \mathcal{H}^n_\delta(A) = \mathcal{H}^n(A)$ for each $A \subset \mathbb{R}^n$, $0 < \delta \leqslant \infty$.
- 11. Theorem (Area formula for injective maps): let $n \leqslant m$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an injective Lipschitz function and $A \subset \mathbb{R}^n$ be a measurable set. Then
$$\mathcal{H}^n(f(A)) = \int_A \mathbf{J}f \, d\mathcal{L}^n$$
and $\mathcal{H}^n \llcorner f(\mathbb{R}^n)$ is a Radon measure on $\mathbb{R}^m$.
- 12. Theorem (Existence of approximate tangent spaces): let
$$\Phi_{x,r} : \mathbb{R}^n \to \mathbb{R}^n, \ \Phi_{x,r}(y) := \frac{y-x}{r}, \ y \in \mathbb{R}^n .$$
If $\Gamma \subset \mathbb{R}^n$ is a locally $\mathcal{H}^k$-rectifiable set, then for $\mathcal{H}^k$-a.e. $x \in \Gamma$ there exists a unique k- dimensional plane π_x such that, as $r \to 0^+$,
$$\frac{(\Phi_{x,r})_{\#}\mu}{r^k} = \mathcal{H}^k \llcorner \left(\frac{\Gamma - x}{r}\right) \stackrel{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x ,$$
that is
$$\lim_{r\to 0^+} \frac{1}{r^k} \int_\Gamma \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^k(y) = \int_{\pi_x} \varphi(y)\, d\mathcal{H}^k(y) \quad \forall \varphi \in \mathbf{C}^0_c(\mathbb{R}^n) .$$
In particular
$$\exists\, \Theta_k(\Gamma, x) := \lim_{r\to 0^+} \frac{\mathcal{H}^k(\Gamma \cap B(x,r))}{\alpha^*_k \, r^k} = 1 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in \Gamma .$$
- 13. Theorem (Rectifiability criterion): if $\Gamma \subset \mathbb{R}^n$ is a compact set, π is a k-dimensional plane in $\mathbb{R}^n$, and there exist δ and t positive with
$$\Gamma \cap B(x,\delta) \subset x + K(\pi, t) \quad \forall x \in \Gamma ,$$
then Γ is $\mathcal{H}^k$-rectifiable, since there exist finitely many Lipschitz maps $f_h : \mathbb{R}^k \to \mathbb{R}^n$ $(h = 1, \dots, N)$ and compact sets $F_h \subset \mathbb{R}^k$ with
$$\Gamma = \cup_{h=1}^N f_h(F_h) .$$
- 14. Theorem (Rectifiability by convergence of the blow-ups): If μ is a Radon measure on $\mathbb{R}^n$, Γ is a Borel set in $\mathbb{R}^n$, μ is concentrated on Γ (that is $\mu = \mu \llcorner \Gamma$), and, for every $x \in \Gamma$, there exists a k-dimensional plane π_x in $\mathbb{R}^n$ such that
$$\frac{(\Phi_{x,r})_{\#}\mu}{r^k} \stackrel{*}{\rightharpoonup} \mathcal{H}^k \llcorner \pi_x \text{ as } r \to 0^+,$$
then $\mu = \mathcal{H}^k \llcorner \Gamma$ and Γ is locally $\mathcal{H}^k$-rectifiable.

Final examination procedure : the final examination will be an interview with the student. Before the interview, the student must agree with the teacher two results fron the above list, with their proofs.Then the student will write a report on those results, which must be sent to the teacher two/three days before the interview. The main part of the interview will focus on this report. A second part will deal with some notions and results related to the first one, but without proofs.

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