ADVANCED ANALYSIS
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Francesco Serra Cassano

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($\star$)= optional argument

($\circ$)= result without proof

I. Derivation of measures and functions.

I.1 Some recalls of measure theory: measures and outer measures, Lebesgue-Stieltjes measure, approximation of measures.

Recalls about the classical fundamental theorem of calculus.

Definitions of outer measure φ on a set X and φ-measurability (or Carathéodory measurability). Definition of the class of φ-measurable sets in X, $\mathcal{M}_\varphi$: $\mathcal{M}_\varphi$ is a σ-algebra and the set function $\varphi : \mathcal{M}_\varphi \to [0, \infty]$ is countable additive, left and right continuous($\circ$).

Definition of Carathéodory outer measure on a metric space. Carathéodory's criterion($\circ$): if φ is a Carathéodory outer measure on a metric space X, then all closed and open sets are φ-measurable. In particulay, $\mathcal{B}(X) \subset \mathcal{M}_\varphi$. Definition of the Borel σ-algebra, $\mathcal{B}(X)$, on a topological space X.

Definitions, on a topological space X, of Borel, Borel regular and Radon outer measure.

Definition of measure $\mu : \mathcal{M} \to [0, \infty]$, where $\mathcal{M}$ is a σ-algebra of a set X. Definition of mesaure space $(X, \mathcal{M}, \mu)$. Definitions, in a measure space $(X, \mathcal{M})$, of Borel, Radon, finite and σ-finite measure $\mu : \mathcal{M} \to [0, \infty]$. A measure $\mu : \mathcal{M} \to [0, \infty]$ is monotone, countably additive , right and left continuous.

Definition of outer measure μ^* generated by a measure μ defined on a measure space $(X, \mathcal{M})$.

Carathéodory-Hahn extension theorem($\circ$): let $(X, \mathcal{M}, \mu)$ be a measure space. and let μ^* denote the outer measure generated by μ. Then

(i) $\mathcal{M}_{\mu^*} \supset \mathcal{M}$ and $\mu^* = \mu$ on $\mathcal{M}$.

(ii) Let $\mathcal{N}$ be a σ-algebra with $\mathcal{M} \subset \mathcal{N} \subset \mathcal{M}_{\mu^*}$ and suppose ν is a measure on $\mathcal{N}$ such that $\nu = \mu$ on $\mathcal{M}$. Then $\nu = \mu^*$ on $\mathcal{N}$, provided μ is σ-finite.

Approximation of an outer measure by closed an open sets : let φ be a Borel (respectively a Borel regular) outer measure on a metric space X. Suppose there exist a sequence of open sets $(V_i)_i \subset X$ such that $X = \cup_{i=1}^{\infty} V_i$ with $\varphi(V_i) < \infty \, \forall i$. Then, for each $E \in \mathcal{B}(X)$ (respectively $\in \mathcal{M}_\varphi$),

$\varphi(E) = \inf\{\varphi(U) : U \supset E, U \,\text{open}\}$; $\varphi(E) = \sup\{\varphi(C) : C \subset E, C \,\text{closed}\}$.

Approximation by compact, closed and open sets for a Borel measure: consider a measure space $(X, \mathcal{B}(X), \mu)$ where X is a metric space and suppose there exists a sequence of open sets $(V_i)_i \subset X$ such that $X = \cup_{i=1}^{\infty} V_i = X$ with $\mu(V_i) < \infty \, \forall i$. Then, for each $E \in \mathcal{B}(X)$,

(i) $\mu(E) = \inf\{\mu(U) : \, U \,\text{open}, U \supset E\}$;

(ii) $\mu(E) = \sup\{\mu(C) : \, C \,\text{closed}, C \subset E\}$.

Moreover, if X is also a separable, locally compact metric space and μ is a Radon measure, then it also holds $\mu(E) = \sup\{\mu(K) : \, K \,\text{compact}, K \subset E\}(\circ)$.

Definition of Lebesgue-Stieltjes measure λ_f on $\mathbb{R}$, induced by a nondecreasing function $f : \mathbb{R} \to \mathbb{R}$. The set function λ_f is a Radon outer measure on $\mathbb{R}$. $\lambda_f((a, b]) = f(b) - f(a)$ for all $a, b \in \mathbb{R}$ with $a < b$, provided $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing and right-continuous ($\circ$). Let μ be a finite Borel outer measure on $\mathbb{R}$ and let $f(x) := \mu((-\infty, x]) \quad \forall x \in \mathbb{R}$. Then $\lambda_f(B) = \mu(B) \quad \forall B \in \mathcal{B}(\mathbb{R})$.

Fundamental theorem of calculus in terms of measures.

I.2 The Radon- Nikodym and Lebesgue decomposition theorems.

Definitions of absolutely continuous and mutually singular measures in a measure space $(X, \mathcal{M})$.

A characterization of the absolutely continuity($\circ$): let ν be a finite measure and μ a measure on a measure space $(X, \mathcal{M})$. Then the following are equivalent: (i) $\nu << \mu$; (ii) $\lim_{\mu(A)\to 0} \nu(A) = 0$, that is, for every $\varepsilon > 0 \; \exists \delta = \delta(\varepsilon) > 0$ such that $\nu(E) < \varepsilon$ whenever $\mu(E) < \delta$.

Radon-Nikodym's theorem: let ν and μ be two measures on $(X, \mathcal{M})$. Suppose that ν and μ are σ-finite and $\nu << \mu$. Then there exists a measurable function $w : X \to [0, \infty]$, called the Radon-Nikodym derivative of ν with respect to μ and denoted by $w = \dfrac{d\nu}{d\mu}$, such that $\nu = \mu_w$ on $\mathcal{M}$, that is, $\nu(E) = \mu_w(E) := \int_E w \, d\mu \quad \forall E \in \mathcal{M}$.

When μ is not σ-finite, the Radon-Nikodym theorem fails($\star$). If X is supposed to be a locally compact, separable metric space and ν and μ are Radon measures with $\nu << \mu$, then the Radon-Nikodym derivative $w := \frac{d\nu}{d\mu}$ is locally integrable.

Lebesgue's decomposition theorem: let ν and μ be σ-finite measures on a measure space $(X, \mathcal{M})$. Then there is a decomposition of ν such that $\nu = \nu_{ac} + \nu_s$ with $\nu_{ac} << \mu$ and ν_s and μ mutually singular. The decomposition is unique.

Definitions of signed measure, a signed measure absolutely continuous with respect a measure, σ-finite signed measure. Radon- Nikodym's theorem for signed measures($\star$).

I.3 Lebesgue points and differentiation theorem for Radon measures on $\mathbb{R}^n$: Lebesgue's differentiation theorem for monotone functions.

Lebesgue's differentiation theorem($\circ$): let $f \in L^1_{loc}(\mathbb{R}^n)$, then there exists $\lim_{r\to 0} \dfrac{\int_{B(x,r)} f(y)\,dy}{|B(x,r)|} = f(x)$ for a.e.$x \in \mathbb{R}^n$, where $|B(x,r)| := \mathcal{L}^n(B(x,r))$.

Lebesgue points' theorem: let $f \in L^1_{loc}(\mathbb{R}^n)$, then a.e. $x \in \mathbb{R}^n$ is a Lebesgue point of f, that is, there exists $\lim_{r\to 0} \dfrac{\int_{B(x,r)} |f(y) - f(x)|\,dy}{|B(x,r)|} = 0\,.$

Vitali covering lemma($\circ$): let $\mathcal{G}$ be a family of closed balls in $\mathbb{R}^n$ such that $R := \sup\{\mathrm{diam}(B) : B \in \mathcal{G}\} < \infty$. Then there exists a countable subfamily $\mathcal{F} \subset \mathcal{G}$ of pairwise disjoint elements such that $\cup_{B\in\mathcal{G}} B \subset \cup_{B\in\mathcal{F}} \hat{B}$, where $\hat{B} = B(x, 5\,r)$ if $B = B(x,r)$,

Derivative of a Radon measure with respect to $\mathcal{L}^n$: let ν be a Radon measure on $\mathbb{R}^n$, then $D_{\mathcal{L}^n}\lambda(x) := \lim_{r\to 0} \dfrac{\nu(B(x,r))}{|B(x,r)|} = \dfrac{d\nu_{ac}}{d\mathcal{L}^n}(x)$ a.e. $x \in \mathbb{R}^n$.

Definition of regular differentiation basis for the Lebesgue measure $\mathcal{L}^n$. Differentiation of a measure with respect to a regular differentiation basis: let ν be a Radon measure on $\mathbb{R}^n$, then there exists $\lim_{h\to\infty} \dfrac{\nu(E_h(x))}{|E_h(x)|} = \dfrac{d\nu_{ac}}{d\mathcal{L}^n}(x)$ a.e. $x \in \mathbb{R}^n$, whenever $(E_h(x))_h$ is a regular differentiation basis of $\mathcal{L}^n$ at x.

Lebesgue's differentiation theorem for monotone functions: let $f : [a,b] \to \mathbb{R}$ be non decreasing. Then (i) there exists $f'(x)$ for a.e. $x \in [a,b]$ and (ii) $\int_a^b f'(x)\,dx \leqslant f(b) - f(a)$. Cantor-Lebesgue- Vitali's function and its properties.

Derivative of the indefinite integral: suppose $f : [a,b] \to \mathbb{R}$ is a Lebesgue integrable function, i.e. f is Lebsegue measurable and $\int_a^b |f|\,dt < \infty$. For each $x \in [a,b]$ let $F(x) := \int_a^x f(t)\,dt$. Then

(i)$F \in \mathbf{C}^0([a,b])$;

(ii)$\exists\, F'(x) = f(x)$ a.e. $x \in [a,b]$.

I.4 Functions of bounded variation. Definition of variation for a function $f : [a,b] \to \mathbb{R}$ and the space $BV([a,b])$ of functions with bounded variation. $BV([a,b])$ is a vector space (on $\mathbb{R}$).

Jordan's decomposition theorem: Let $f : [a,b] \to \mathbb{R}$. Then the following are equivalent: (i) $f \in BV([a,b])$; (ii) there exist $g, h : [a,b] \to \mathbb{R}$ nondecreasing such that $f = g - h$.

I.5 The fundamental theorem of calculus. Characterization of the integral identity by the absolute continuity of measures: let $f : \mathbb{R} \to \mathbb{R}$ be non decreasing and bounded, then the following conditions are quivalent: (i) $f(y) - f(x) = \int_x^y f'(t)\,dt \quad x, y \in \mathbb{R}, x < y$; (ii) f is right-continuous and $\lambda_f << \mathcal{L}^1$.

Definition of the space $AC([a,b])$ of absolutely continuous functions.

Fundamental theorem of calculus for monotone functions: let $f : [a,b] \to \mathbb{R}$ be a nondecreasing function, then the following are equivalent:(i) $f(x) - f(a) = \int_a^x f(t)\,dt \quad \forall x \in [a,b]$; (ii) $f \in AC([a,b])$. A first consequence: let $f : [a,b] \to \mathbb{R}$ be a nondecreasing function. Suppose that $f \in AC([a,b])$ and $f' = 0$ a.e. in $[a,b]$. Then f is constant.

Decomposition theorem in $AC([a,b])$: let $f \in AC([a,b])$, then there exist two nondecreasing, absolutely continuous functions g, $h : [a,b] \to \mathbb{R}$ such that $f = g - h \quad \text{in}[a,b]$.

Fundamental theorem of calculus: let $f : [a,b] \to \mathbb{R}$, then $f \in AC([a,b])$ iff (i) f is differentiable a.e. in $[a,b]$, (ii) f' is integrable in $[a,b]$ and (iii) $f(x) - f(a) = \int_a^x f'(t)\,dt \quad \forall x \in [a,b]$.

II. Main spaces of functions and results on Banach and Hilbert spaces.

II.1 The space of continuous functions $C^0(\overline{\Omega})$. Definition of $C^0(\overline{\Omega})$ when $\Omega \subset \mathbb{R}^n$ is a bounded open set. Norm in $C^0(\overline{\Omega})$: the norm $\|\cdot\|_\infty$. $(C^0(\overline{\Omega}), \|\cdot\|_\infty)$ is an infinite dimensional Banach space which is not a Hilbert space.

Riesz's theorem ($\circ$): let $(E, \|\cdot\|)$ be a normed vector space and denote $B_E := \{x \in E : \|x\| \leqslant 1\}$. Then B_E is compact iff $\dim_{\mathbb{R}} E < \infty$.

Definition of equicontinuous family of functions $\mathcal{F} \subset C^0(A)$ on set $A \subset \mathbb{R}^n$. Arzelà -Ascoli's theorem: let $K \subset \mathbb{R}^n$ be a compact set and let $\mathcal{F} \subset C^0(K)$. Then $\mathcal{F}$ is compact in $(C^0(K), \|\cdot\|_\infty)$ iff (i) $\mathcal{F}$ is bounded in $(C^0(K), \|\cdot\|_\infty)$, (ii) $\mathcal{F}$ is closed in $(C^0(K), \|\cdot\|_\infty)$ and (iii) $\mathcal{F}$ is equicontinuous.

Two consequences of Arzelà - Ascoli's theorem:

- Let $K \subset \mathbb{R}^n$ be a compact set and let $\mathcal{F} \subset C^0(K)$ be bounded and equicontinuous, then $\overline{\mathcal{F}}$ is compact in $(C^0(K), \|\cdot\|_\infty)$.
- Let $f_h : [a,b] \to \mathbb{R}$ $(h = 2, \dots)$ be a continuous sequence of functions. Suppose that: (i) there exists $M > 0$ such that $|f_h(x)| \leqslant M$ for all $x \in [a,b]$, $h \in \mathbb{N}$;(ii) $(f_h)_h$ is equicontinuous on $[a,b]$.Then there exist a subsequence $(f_{h_k})_k$ and a function $f \in C^0[a,b]$ such that $f_{h_k} \to f$ uniformly in $[a,b]$.

Weierstrass' approximation theorem($\circ$): the set of polynomial functions with real coefficients is dense in $(C^0([a,b]), \|\cdot\|_\infty)$. $(C^0(K), \|\cdot\|_\infty)$ is separable, provided $K \subset \mathbb{R}^n$ is compact (proof only in the case $n = 1$ and $K = [a,b]$.)

II.2 The space of continuously differentiable functions $C^1(\overline{\Omega})$. Definition of the space $C^1(\overline{\Omega})$ when $\Omega \subset \mathbb{R}^n$ is a bounded open set. Norm in $C^1(\overline{\Omega})$: the norm $\|\cdot\|_{C^1}$. $(C^1(\overline{\Omega}), \|\cdot\|_{C^1})$ is an infinite dimensional Banach space, which is not a Hilbert space, provided $\Omega \subset \mathbb{R}^n$ is a bounded open set.

Compactness in $(C^1(\overline{\Omega}), \|\cdot\|_{C^1})$: let $\mathcal{F} \subset C^1(\overline{\Omega})$ and denote by $\mathcal{F}_i := \{D_i f : f \in \mathcal{F}\} \quad i = 1, \dots, n$. Then $\mathcal{F}$ is compact in $(C^1(\overline{\Omega}), \|\cdot\|_{C^1})$ iff (i) $\mathcal{F}$ and $\mathcal{F}_i$ are bounded in $(C^0(\overline{\Omega}), \|\cdot\|_{C^0})$, (ii) $\mathcal{F}$ and $\mathcal{F}_i$ are closed in $(C^0(\overline{\Omega}), \|\cdot\|_{C^0})$ and (iii) $\mathcal{F}$ and $\mathcal{F}_i$ are equicontinuous on $\overline{\Omega}$. (Proof only when $n = 1$ and $\Omega = (a,b)$.

$(C^1(\overline{\Omega}), \|\cdot\|_{C^1})$ is separable.

II.3 The space of Lipschitz functions $Lip(\Omega)$. Definition of Lipschitz function $f : A \subset \mathbb{R}^n \to \mathbb{R}$: Lipschitz constant of f, $Lip(f, A)$; the space $Lip(A)$ of Lipschitz functions on $A \subset \mathbb{R}^n$. Definition of Lipschitz function $f : A \subset (X, d_X) \to (Y, d_Y)$ when (X, d_X) and (Y, d_Y) are metric spaces.

Extension of Lipschitz functions: let $A \subset \mathbb{R}^n$ and let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then there exist a unique Lipschitz extension $\bar{f} : \overline{A} \to \mathbb{R}$ with $Lip(\bar{f}, \overline{A}) = Lip(f, A)$, where $\overline{A}$ denote the closure of A. As a consequence $Lip(A) \equiv Lip(\overline{A})$.

Norm in $Lip(\Omega)$: the norm $\|\cdot\|_{Lip}$. $(Lip(\Omega), \|\cdot\|_{Lip})$ is an infinite dimensional Banach space which is not a Hilbert space, provided $\Omega \subset \mathbb{R}^n$ is a bounded open set. $C^1(\overline{\Omega}) \subset Lip(\Omega)$, the inclusion is an isometry and it is strict, provided $\Omega \subset \mathbb{R}^n$ is a bounded open convex set (proof only when $n = 1$ and $\Omega = (a, b)$).

Compactness in $Lip(\Omega)$: let Ω be a bounded open set, then $B_{Lip(\Omega)}$ is compact in $(Lip(\Omega), \|\cdot\|_\infty)$.

$(Lip((a, b)), \|\cdot\|_{Lip})$ is not separable.

Definition of the space of Hölder continuous functions on a subset $A \subset \mathbb{R}^n(\star)$.

II.4. The space of p-integrable functions $L^p(\Omega)$. Definition of the space $L^p(\Omega)$, on an open set $\Omega \subset \mathbb{R}^n$, with respect to the n-dimensional Lebesgue measure. Norm in $L^p(\Omega)$: the norm $\|\cdot\|_{L^p}$.

Riesz-Fisher theorem($\circ$): $(L^p(A), \|\cdot\|_{L^p})$ is a B.s. if $1 \leqslant p \leqslant \infty$. Moreover $L^2(A)$ turns out to be a Hilbert space with respect to the scalar product $(f, g)_{L^2} := \int_A f\, g\, dx \quad f,\, g \in L^2(A)$.

As a consequence of the proof of Riesz-Fisher's theorem: let $1 \leqslant p \leqslant \infty$ and let $(f_h)_h \subset L^p(\Omega)$ be a sequence such that there exists $f \in L^p(\Omega)$ for which $f_h \to f$ in $L^p(\Omega)$. Then there exists a subsequence $(f_{h_i})_i$ of $(f_h)_h$ such that $f(x) = \lim_{i\to\infty} f_{h_i}(x)$ $\mu-$a.e.$x \in \Omega$.

Compactness in $(L^p(\Omega), \|\cdot\|_{L^p})(\star)$:

- M. Riesz- Fréchét-Kolmogorov 's theorem($\circ$): Let $\mathcal{F}$ be a <u>bounded</u> subset in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ with $1 \leqslant p < \infty$. Suppose that $\lim_{v\to 0} \|\tau_v f - f\|_{L^p} = 0$ uniformly for $f \in \mathcal{F}$, that is

$$(EN_{\mathcal{F}}) \qquad \forall\, \epsilon > 0\, \exists\, \delta = \delta(\epsilon) > 0 \text{ such that } \|\tau_v f - f\|_{L^p} < \epsilon \quad \forall\, v \in \mathbb{R}^n \text{ with } |v| < \delta,\ \forall\, f \in \mathcal{F}\,.$$

  Then $\mathcal{F}|_\Omega := \{f|_\Omega : f \in \mathcal{F}\}$ is relatively compact in $(L^p(\Omega), \|\cdot\|_{L^p})$, i.e. its closure is compact in $(L^p(\Omega), \|\cdot\|_{L^p})$, for each open set $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure.
- Characterization of compactness in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$: let $\mathcal{F} \subset L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$. Then $\mathcal{F}$ is relatively compact in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$ if and only if
  (i) $\mathcal{F}$ is bounded in $(L^p(\mathbb{R}^n), \|\cdot\|_{L^p})$;
  (ii) for each $\epsilon > 0$ there exists $r_\epsilon > 0$ such that
  $$\|f\|_{L^p(\mathbb{R}^n \setminus B(0, r_\epsilon))} < \epsilon \quad \forall\, f \in \mathcal{F}\,;$$
  (iii) $\lim_{v\to 0} \|\tau_v f - f\|_{L^p} = 0$ uniformly for $f \in \mathcal{F}$.

Urysohn's lemma($\circ$): let X, be a locally compact metric space, let $K \subset X$ and $V \subset X$ be, respectively, a compact set and an open set such that $K \subset V$. Then there exists a function $\varphi \in C^0_c(X)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ in K and $\mathrm{spt}(\varphi) \subset V$.

Approximation by simple function(∘): let $(X,\mathcal{M})$ be a measure space and let $f : X \to [0,+\infty]$ be a measurable function. Then there exists a sequence of measurable simple functions $s_h : X \to [0,+\infty]$ $(h = 1,2,\dots)$ satisfying the properties:

(i) $0 \leqslant s_1 \leqslant s_2 \leqslant \dots \leqslant s_h \leqslant \dots \leqslant f$;
(ii) $\lim_{h\to\infty} s_h(x) = f(x) \quad \forall\, x \in X$:

In particular, if $f \in L^1(X,\mu)$, that is $\int_X f\,d\mu < \infty$, then $s_h \to f$ in $L^1(X,\mu)$, that is,

$$\|f - s_h\|_{L^1(X,\mu)} := \int_X |f - s_h|\,d\mu \to 0\,.$$

Lusin's theorem(∘): let μ be a Radon measure on a locally compact, separable metric space X . Let $f : X \to \overline{\mathbb{R}}$ be a measurable function such that there exists a Borel set $A \subset X$ with $\mu(A) < \infty$, $f(x) = 0 \quad \forall\, x \in X\setminus A$ and $|f(x)| < \infty \quad \mu-$a.e. $x \in X$. Then, for each $\epsilon > 0$, there exists $g \in \mathbf{C}^0_c(X)$ such that $\mu\left(\{x \in X : f(x) \neq g(x)\}\right) < \epsilon$. Moreover g can be chosen such that $\sup_{x\in X}|g(x)| \leqslant \sup_{x\in X}|f(x)|$.

Approximation by continuous functions in L^p: let $\Omega \subset \mathbb{R}^n$ be an open set, then $C^0_c(\Omega)$ is dense in $(L^p(\Omega), \|\cdot\|_{L^p})$, provided that $1 \leqslant p < \infty$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then $(L^p(\Omega), \|\cdot\|_{L^p})$ is separable if $1 \leqslant p < \infty$ and it is not separable if $p = \infty$.

Let $\Omega \subset$ be a bounded open set, then the inclusion $C^0(\overline{\Omega}) \subset L^\infty(\Omega)$ is strict. Moreover, for each $f \in C^0(\overline{\Omega})$ $\|f\|_{\infty,\Omega} = \|f\|_{L^\infty(\Omega)}$ and $C^0(\overline{\Omega})$ is closed in $(L^\infty(\Omega), \|\cdot\|_{L^\infty(\Omega)})$.

Dual space E' of a normed vector space $(E, \|\cdot\|_E)$ and dual norm $\|\cdot\|_{E'}$. $(E', \|\cdot\|_{E'})$ is a Banach space.

Density criterion for subspaces: let $(E, \|\cdot\|_E)$ be a n.v.s, assume that $M \subset E$ is a subspace that is not dense in $(E, \|\cdot\|_E)$ and let $x_0 \in E \setminus \overline{M}$. Then there exists $f \in E'$ such that $\langle f, x\rangle_{E'\times E} = 0 \quad \forall\, x \in \overline{M} \quad$ and $\quad \langle f, x_0\rangle_{E'\times E} = 1$.

If $(E', \|\cdot\|_{E'})$ is separable so is $(E, \|\cdot\|_E)$.

Riesz representation theorem(∘): let $1 \leqslant p < \infty$ and denote $p' = \frac{p}{p-1}$ if $1 < p < \infty$ and $p' = \infty$ if $p = 1$. Then the map $T : L^{p'}(\Omega) \to (L^p(\Omega))'$, defined by $< Tu, f >_{(L^p(\Omega))'\times L^p(\Omega)} := \int_\Omega u\, f\, dx \quad \forall f \in L^p(\Omega)$, is an isomorphism and an isometry

Essential support of a function $f \in L^p(\mathbb{R}^n)$: let $\Omega \subset \mathbb{R}^n$ be an open set. Denote $\mathcal{A}_f := \{\omega \subset \mathbb{R}^n : f = 0 \text{ a.e. in}\,\omega\}$ and let $A_f := \cup_{\omega\in\mathcal{A}_f}\omega$. Then A_f is an open set and $f = 0$ a.e. in A_f . The closed set $\mathrm{spt}_e(f) := \Omega \setminus A_f$ is called the essential support of f in $\mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $\mathbb{R}^n \setminus A_f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.

Definition of a mollifiers sequence $(\varrho_h)_h$, in the sense of Friedrichs, and its construction.

Definition of convolution product $\varrho_h * f$ between a mollifier ϱ_h and a function $f \in L^1_{loc}(\mathbb{R}^n)$: $\varrho_h * f : \mathbb{R}^n \to \mathbb{R}$ is well defined, continuous and $(\varrho_h * f)(x) = (f * \varrho_h)(x)$ for all $x \in \mathbb{R}^n$ and $h \in \mathbb{N}$.

Approximation by convolution in $L^p(\mathbb{R}^n)$: let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(\varrho_h)_h$ be a squence of mollifiers. Then (i) $f * \varrho_h \in C^\infty(\mathbb{R}^n)$ for each $h \in \mathbb{N}$; (ii)$\|f * \varrho_h\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}$ for each $h \in \mathbb{N}$, $f \in L^p(\mathbb{R}^n)$, for every $p \in [1,\infty]$; (iii) $\mathrm{spt}(f * \varrho_h) \subset \mathrm{spt_e}(f) + \overline{B(0,1/h)}$ for each $h \in \mathbb{N}$ and (iv) if $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, then $f * \varrho_h \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for each $h \in \mathbb{N}$, and $f * \varrho_h \to f$ as $h \to \infty$, in $L^p(\mathbb{R}^n)$, provided $1 \leqslant p < \infty$. This results yields the two following results.

Fundamental lemma of Calculus of Variations: let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in L^1_{loc}(\Omega)$. Assume that $\int_\Omega f\,\varphi\,dx = 0 \quad \forall \varphi \in C^\infty_c(\Omega)$. Then $f = 0$ a.e. in Ω.

Approximation by C^∞ functions in $L^p(\Omega)$: let $\Omega \subset \mathbb{R}^n$ be an open set. Then $C^\infty_c(\Omega)$ is dense in $(L^p(\Omega), \|\cdot\|_{L^p})$, provided that $1 \leqslant p < \infty$.

III. Weak topologies.

III.1 Weak topology on a normed vector space and compactness. Definition of strong topology τ_s on a normed vector space $(E, \|\cdot\|)$ and problem of the introduction of a weak topology. Definition of weak convergence for a sequence in a normed vector space and idea for the construction of a convergent subsequence: coordinatewise convergence .

Existence of the weak topology(∘): there exists a topology on a normed vector space $(E, \|\cdot\|)$, denoted by $\sigma(E, E')$, such that

(i) $\sigma(E, E')$ is the smallest topology on E with respect to which each $f \in E'$ is continuous.

(ii) $(E, \sigma(E, E'))$ is a topological Hausdorff space, that is, for each pair of distinct points x_1 and x_2 in E there exist disjoint open sets U_1 and U_2 in $\sigma(E, E')$ with $x_i \in U_i$ $i = 1, 2$.

(iii) Let $(x_h)_h \subset E$. Then

$$x_h \overset{\sigma(E,E')}{\to} x \text{ as } h \to \infty \quad \Longleftrightarrow$$
$$\langle f, x_h\rangle_{E'\times E} \to \langle f, x\rangle_{E'\times E} \text{ as } h \to \infty\,, \forall\, f \in E'\,.$$

$\sigma(E, E') = \tau_s$, if $\dim_{\mathbb{R}} E < \infty$ and $\sigma(E, E')$ is less fine than τ_s, if $\dim_{\mathbb{R}} E = \infty$(∘).

In a normed vector space $(E, \|\cdot\|)$, a convex set $C \subset E$ is closed (with respect to τ_s) iff C is weakly closed (that is, closed with respect to $\sigma(E, E')$).

Comparison between the weak and strong convergence in a normed vector spaces: suppose that $(E, \|\cdot\|)$ is a n.v.s and let $(x_h)_h \subset E$ and $x \in E$. Then the following assertions hold:

(i) if $x_h \to x$, then $x_h \rightharpoonup x$.

(ii) If $x_h \rightharpoonup x$, then $(x_h)_h$ is bounded in $(E, \|\cdot\|)$, that is, the sequence $(\|x_h\|)_h$ is bounded and $\|x\| \leqslant \liminf_{h\to\infty} \|x_h\|$.

(iii) Let $(f_h)_h \subset E'$. Assume that $f_h \to f$ in E' and $x_h \rightharpoonup x$ in E . Then $\langle f_h, x_h\rangle_{E'\times E} \to \langle f, x_h\rangle_{E'\times E}$.

Characterization of the weak convergence in $(L^p(\Omega), ||\cdot||_{L^p})$ if $1 \leqslant p < \infty$.

Definition of reflexive normed vector space. Each Hilbert space is reflexive.

Stability properties of reflexive spaces(∘): (i) if $(E, \|\cdot\|_E)$ is a reflexive Banach space then if $M \subset E$ is a closed vector subspace then $(M, , \|\cdot\|_E)$ is also reflexive; (ii) $(E', \|\cdot\|)$ is reflexive iff $(E, \|\cdot\|_E)$ is reflexive.

Weak sequential compactness of the unit closed ball: B_E is sequentially compact with respect to the topology $\sigma(E, E')$, provided that $(E, \|\cdot\|_E)$ is a reflexive normed vector space. More generally, each closed, convex and bounded set $C \subset E$ is weakly sequentially compact.

III.2 Reflexivity of the main spaces of functions. $(C^0(\bar\Omega), ||\cdot||_\infty)$, $(C^1(\bar\Omega), ||\cdot||_{C^1})$ and $(Lip(\Omega), ||\cdot||_{Lip})$ are not reflexive.

$(L^p(\Omega), ||\cdot||_{L^p})$ is reflexive if $1 < p < \infty$, but $(L^p(\Omega), ||\cdot||_{L^p})$ is not reflexive if $p = 1, \infty$. The unit closed ball $B_{L^p(\Omega)}$ is sequentially compact in $(L^p(\Omega), ||\cdot||_{L^p})$ if $1 < p < \infty$.

III.3 Weak topology and convexity: an application to the Calculus of Variations.

Definitions of convexity and semicontinuity for a function $\varphi : A \to (-\infty, +\infty]$ when $A \subset E$ and E is a vector space equipped with a topology. In a normed vector space $(E, \|\cdot\|)$, a semicontinuous and convex function $\varphi : A \to (-\infty, +\infty]$ is also semicontinuous with respect to the weak topolgy $\sigma(E, E')$, provided $A \subset E$ is convex and closed.

Generalized Weierstrass theorem in reflexive spaces: let $(E, \|\cdot\|)$ be a reflexive normed space and let $\varphi : A \subset E \to (-\infty, +\infty]$. Suppose that (i) A is closed and φ is convex; (ii) A is bounded or A is unbounded but there exists $\lim\limits_{x\in A, \|x\|\to+\infty} \varphi(x) = +\infty$; (iii) φ is semicontinuous (with respect to τ_s). Then there exists $\min\limits_{A} \varphi$.

Two consequences of the generalized Weierstrass theorem in reflexive spaces are

• In a reflexive normed vector space $(E, \|\cdot\|)$, each $f \in E'$ attains its maximun on the unit closed ball B_E and $\|f\|_{E'} = \max\limits_{B_E} f$.

• Projection on a convex set in reflexive spaces: let $(E, \|\cdot\|)$ be a reflexive normed space and let $A \subset E$ be closed and convex. Then for each $x_0 \in E$ there exists $\min\limits_{x\in A} \|x - x_0\|$.

III.4 Weak* topology and compactness. ($\star$)

Definition and existence of the weak* topology: there exists a topology on a dual space $(E', \|\cdot\|_{E'})$, denoted by $\sigma(E', E)$, such that

(i) $\sigma(E', E)$ is the smallest topology on E' with respect to which each $\phi \in J_E(E)$ is continuous.

(ii) $(E', \sigma(E', E))$ is a topological Hausdorff space.

(iii) Let $(f_h)_h$, $f \subset E'$. Then

$$f_h \overset{\sigma(E',E)}{\to} f \text{ as } h \to \infty \iff$$
$$\langle f_h, x\rangle_{E'\times E} \to \langle f, x\rangle_{E'\times E} \text{ as } h \to \infty\,, \forall\, x \in E\,;$$

(iv) If E is reflexive, then $\sigma(E', E'') = \sigma(E', E)$.

Banach-Bourbaki-Alaoglu theorem: let $(E, \|\cdot\|)$ be a v.n.s. Then the closed unit ball $B_{E'} := \{f \in E' : \|f\|_{E'} \leqslant 1\}$ is compact in the weak* topology $\sigma(E', E)$.

Weak* sequential compactness in separable spaces: if $(E, \|\cdot\|)$ is a separable space, then $(B_{E'}, \sigma(E', E))$ is sequentially compact.

An application: let $\Omega \subset \mathbb{R}^n$ be an open set. Then the unit closed ball of $L^\infty(\Omega)$ $B_{L^\infty(\Omega)}$ is weakly* sequentially compact.

IV. An introduction to the Sobolev space and an application to Poisson's equation.

IV.1 Reference example: general electrostatic problem. Gauss' law, Poisson's equations and boundary-value (or Dirichlet) problem for Poisson's equation in a bounded open set $\Omega \subset \mathbb{R}^3$, well-posedeness and classical solutions.

IV.2 Energy functional and classical Dirichlet's principle. Extension of the boundary-value problem for Poisson's equation to any dimension: problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma \end{cases},$$

for given $f : \Omega \to \mathbb{R}$, $g : \Gamma \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ bounded open set and $\Gamma = \partial\Omega$.

Energy functional associated to problem (P): $I : \mathcal{A} \to \mathbb{R}$, $\quad I(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f\,v\,dx \quad v \in \mathcal{A}$, where $\mathcal{A} := \left\{v \in \mathbf{C}^2(\overline{\Omega}) : v = g \text{ on } \Gamma\right\}$ (class of admissible functions).

Dirichlet's principle: let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\Gamma = \partial\Omega$ of class $\mathbf{C}^1$ and let $f \in \mathbf{C}^0(\overline{\Omega})$ and $g \in \mathbf{C}^0(\Gamma)$. Assume $u \in \mathcal{A}$ solves the boundary-value problem (P). Then

$$\text{(DP)} \qquad I(u) = \min_{v\in\mathcal{A}} I(v)\,.$$

Viceversa, if $u \in \mathcal{A}$ satisfies (DP), then u solves the boundary-value problem (P).

Uniqueness of boundary-value problem (P): let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\Gamma = \partial\Omega$ of class $\mathbf{C}^1$ and let $f \in \mathbf{C}^0(\overline{\Omega})$ and $g \in \mathbf{C}^0(\Gamma)$. There exists at most one solution $u \in \mathbf{C}^2(\overline{\Omega})$ of boundary-value problem (P).

IV.3 Dirichlet's principle in the Sobolev space and weak solutions. Homogeneous boundary-value problem for Poisson's equation: problem

$$(P_0) \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases},$$

Definition of Sobolev space $H^1_0(\Omega)$ and its characterization: the following are equivalent: (i) $u \in H^1_0(\Omega)$; (ii) $u \in L^2(\Omega)$ and there exist a sequence $(u_h)_h \subset \mathcal{A}$ and $w \in (L^2(\Omega))^n$ such that $u_h \to u \;$ in $L^2(\Omega) \;$ and $\; \nabla u_h \to w \;$ in $(L^2(\Omega))^n$. Moreover the vector $w = (w_1, \dots, w_n) \in (L^2(\Omega))^n$ in the statement (ii) satisfies the following integration by parts identity

$$\text{(IP)} \qquad \int_\Omega u\,\frac{\partial\varphi}{\partial x_i}\,dx = -\int_\Omega w_i\,\varphi\,dx \quad \forall\,\varphi \in \mathbf{C}^\infty_c(\Omega),\ \forall\, i = 1, \dots, n\,.$$

In particular the vector w is uniquely defined a.e. in Ω.

Definition of weak gradient $Du \in (L^1_{loc}(\Omega))^n$ for a function $u \in L^1_{loc}(\Omega)$.

Poincaré inequality($\circ$): there exists a positive constant $C = C(\Omega) > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C\,\|Du\|_{(L^2(\Omega))^n} \quad \forall\, u \in H^1_0(\Omega)\,.$$

$(H^1_0(\Omega), (\cdot,\cdot)_{H^1_0})$ is a H.s. where $(u,v)_{H^1_0} := (Du, Dv)_{(L^2(\Omega))^n}$ if $u, v \in H^1_0(\Omega)$. In particular the inclusion mapping $i : H^1_0(\Omega) \to L^2(\Omega)$, $\quad i(u) := u$, is continuous.

Dirichlet's principle in the Sobolev space: let $f \in L^2(\Omega)$ and consider the Dirichlet energy functional $I : H^1_0(\Omega) \to \mathbb{R}$, $\quad I(v) := \frac{1}{2}\int_\Omega |Dv|^2\,dx - \int_\Omega f\,v\,dx \quad v \in H^1_0(\Omega)$. Then there exists a unique $u \in H^1_0(\Omega)$ such that

$$\text{(DP)} \qquad I(u) = \min_{H^1_0(\Omega)} I\,.$$

Moreover u is characterized by the following property: $u \in H^1_0(\Omega)$ is the unique solution of the Euler-Lagrange equation, in weak form,

$$\text{(EL)} \qquad \int_\Omega (Du, Dv)_{\mathbb{R}^n}\,dx = \int_\Omega f\,v\,dx \quad \forall\, v \in H^1_0(\Omega)\,.$$

Definition of weak solution for the homogeneous boundary-value problem associated to Poisson's equation.

Definition of higher Sobolev spaces $H^m(\Omega)(\star)$.

Characterization of the 1-dimensional Sobolev space.

Regularity of weak solutions($\star$): let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that

(i) $\Gamma = \partial\Omega$ is of class $\mathbf{C}^{m+2}$ with $m > \frac{n}{2}$;
(ii) $f \in H^m(\Omega)$;
(iii) $u \in H_0^1(\Omega)$ is a weak solution of problem (P_0).

Then $u \in C^2(\overline{\Omega})$ and $u(x) = 0$ for each $x \in \Omega$. In particular, u is a classical solution of problem (P_0).

Exercises that will be the content of the final written test

(*)= demanding exercise , (**)= very demanding exercise

I. Derivation of measures and functions.

I.1 (Dirichlet function) Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be $f(x) := \chi_{\mathbb{Q}}(x)$. Prove that
(i) f is Lebesgue measurable;
(ii) f is not Riemann integrable on any interval $[a, b]$, for each $-\infty < a < b < +\infty$;
(iii) f is discontinuous at every point $x \in \mathbb{R}$;
(iv) Let us consider $X = [0, 1]$ as the topological space endowed with the classical Euclidean topology and consider $f : (X, \mathcal{M}_1 \cap [0, 1]) \to \overline{\mathbb{R}}$, where by $\mathcal{M}_1 \cap [0, 1]$ we denote the σ-algebra of Lebesgue measurable sets of $\mathbb{R}$ contained in $[0, 1]$. Suppose $\mathbb{Q} \cap [0, 1] = \{q_i : i = 1, 2, \dots\}$. For given $\varepsilon > 0$, let $F := [0, 1] \setminus \cup_{i=1}^{\infty}(q_i - 2^{-(i+1)}\varepsilon, q_i + 2^{-(i+1)}\varepsilon)$. Prove that f is measurable, F is closed, $f \equiv 0$ on F and $Ł^1([0, 1] \setminus F) < \varepsilon$.
Remark: Notice that f is continuous on F with respect to the relative topology on F induced by the topology of $[0, 1]$, even though f is discontinuous at every point of F with respect to the topology of $[0, 1]$.

I.2 Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function and let λ_f denote the Lebesgue-Stieltjes outer measure on $\mathbb{R}$ induced by f. Prove that
(i) $\forall E \subset \mathbb{R}\, \exists$ a Borel set $B \subset \mathbb{R}$ such that $E \subset B$ and $\lambda_f(E) = \lambda_f(B)$;
(ii) for each compact set $K \subset \mathbb{R}$ it holds that $\lambda_f(K) < +\infty$.

Remark: From exercise I.2 it follows that λ_f is a Radon outer measure on $\mathbb{R}$.
(**Hint:** (i) If $\lambda_f(E) = +\infty$ choose $B = \mathbb{R}$. If $\lambda_f(E) < +\infty$, for each $k \in \mathbb{N}$ there exists a family $\{I_h^{(k)} : h \in \mathbb{N}\}$, $I_h^{(k)} = (a_h^{(k)}, b_h^{(k)}]$ such that $E \subset \cup_{h=1}^{\infty} I_h^{(k)}$ and $\lambda_f(E) \leqslant \sum_{h=1}^{\infty} \alpha_f(I_h^{(k)}) \leqslant \lambda_f(E) + \frac{1}{k}$. By choosing $B := \cap_{k=1}^{\infty}(\cup_{h=1}^{\infty} I_h^{(k)})$ we obtain the desired assertion.
(ii) There exist $a, b \in \mathbb{R}$ con $a < b$ such that $K \subset (a, b]$. Then $\lambda_f(K) \leqslant \alpha_f((a, b]) < +\infty$.)

I.3 Let $f(x) := x$ if $x \in \mathbb{R}$. Then $\lambda_f(E) = \mathcal{L}^1(E)$ for any $E \subset \mathbb{R}$.
(**Hint:** By definition, it is immediate that $\mathcal{L}^1(E) \leqslant \lambda_f(E)$. To prove the opposite inequality, observe that if $\cup_{i=1}^{\infty}[a_i, b_i] \supset E$, then $\cup_{i=1}^{\infty}(a_i - \varepsilon\, 2^{-i}, , b_i] \supset E$ and $\lambda_f(E) \leqslant \sum_{i=1}^{\infty}(b_i - a_i) + \varepsilon$ for each $\varepsilon > 0$.)

I.4 Let μ be a Borel measure (or also a Borel outer measure) finite on $\mathbb{R}$ (that is $\mu(\mathbb{R}) < +\infty$). Define $f(x) := \mu((-\infty, x])$ if $x \in \mathbb{R}$. Prove that
(i) $f : \mathbb{R} \to \mathbb{R}$ è is nondecreasing and right-continuous;
(ii) f is continuous at a point x_0 iff $\mu(\{x_0\}) = 0$;
(iii) $\mu((a, b]) = f(b) - f(a)$ for each a, $b \in \mathbb{R}$.
(**Hint:** (i) To prove that f is right-continuous at x observe that if $(x_h)_h \subset \mathbb{R}$ is a decreasing sequence, converging to x, then

$$\lim_{h\to\infty} f(x_h) = \lim_{h\to\infty} \mu((-\infty, x_h]) = \mu(\cap_{h=1}^{\infty}(-\infty, x_h]) = f(x)\,.)$$

I.5 Let φ be an outer measure on a set X. Let $(E_i)_i$ be a sequence of subsets of X. Then there exists a sequence of disjoint sets $(A_i)_i$ such that $A_i \subset E_i$ and $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} A_i$. In case the sets E_i, for each $i \in \mathbb{N}$, are φ-measurable, so are A_i.
(**Hint:**For each $j \in \mathbb{N}$, define $S_j = \cup_{i=1}^{j} E_j$. Note that

$$\cup_{i=1}^{\infty} E_i = S_1 \cup (\cup_{k=1}^{\infty}(S_{k+1} \setminus S_k))\,.$$

Now take $A_1 := S_1$ and $A_{i+1} := S_{i+1} \setminus S_i$ for all $i \geqslant 1$.)

I.5.1 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $w_i : X \to [0, +\infty]$ $(i = 1, 2)$ be two measurable functions such that $\displaystyle\int_E w_1\, d\mu = \int_E w_2\, d\mu$ for each $E \in \mathcal{M}$. Then $w_1 = w_2$ μ-a.e. in X.

I.5.2 Let ν and μ be two σ-finite measures on $(X, \mathcal{M})$ and suppose that (RN) holds, that is, $w = \dfrac{d\nu}{d\mu}$. Then, for each non negative measurable function g, it holds that

$$\int_X g\, d\nu = \int_X g\, \frac{d\nu}{d\mu}\, d\mu\,.$$

I.6 Let $U \subset \mathbb{R}$ be an open set. Then U is a countable union of disjoint open intervals.
(**Hint:** For each $x \in U$, let I_x denote the biggest open interval containing x and contained in U. Prove that $U = \cup_{x\in U} I_x$ and $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Also prove that the the family $\{\, I_x : x \in U \,\}$ is countable.)

I.7 Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing function. Prove that f has at most a countable set of discontinuity points.

I.8 Let $X = (0, 1)$, $\nu = \mathcal{L}^1$, $\mu = \#$ and $\mathcal{M} = \mathcal{M}_1 \cap (0, 1)$. Prove that:
(i) μ is not σ-finite;
(ii) $\nu << \mu$;
(iii) there is no a measurable function $w : (0, 1) \to [0, \infty]$ such that $\nu(E) = \int_E w\, d\mu$ for each $E \in \mathcal{M}$.
(**Hint:** (iii) First prove that, for given $x \in (0, 1)$, $\displaystyle\int_{\{x\}} w\, d\mu = w(x)$ for any measurable function $w : (0, 1) \to [0, \infty]$. Then conclude by contradiction.)

I.9 Let λ and ν be two Radon measures on $\mathbb{R}$. Suppose that $\lambda(\mathbb{R}) < +\infty$. Then the following are equivalent:
(i) $\lambda << \nu$;

(ii) for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{m} \lambda((a_i, b_i)) < \varepsilon \,,$$

for each disjoint family of intervals $(a_1, b_1), \ldots, (a_m, b_m)$, verifying

$$\sum_{i=1}^{m} \nu((a_i, b_i)) < \delta \,.$$

(**Hint:** Recall that $\lambda << \nu$ if and only if

(AC)
$$\forall \varepsilon > 0 \, \exists \, \delta = \delta(\varepsilon) > 0 \quad \text{such that} \quad \forall \, A \in \mathcal{B}(\mathbb{R}) \, \text{with} \, \nu(A) < \delta \Rightarrow \quad \lambda(A) < \varepsilon \,.$$

From (AC) it follows the implication $(i) \Rightarrow (ii)$. To prove the reverse implication, use again (AC). Recall that, for each $A \in \mathcal{B}(\mathbb{R})$ with $\nu(A) < \delta$, there exists an open set $U \supset A$ tale che $\nu(U) < \delta$. Then apply exercise I.6.)

I.10 Let consider $\mu = \mathcal{L}^1$, $\nu = \delta_0$ as measures on the σ- algebra $\mathcal{M}_1$ of Lebesgue measurable sets in $\mathbb{R}$, where δ_0 denotes the Dirac measure concetrated at 0, defined by $\delta_0(E) := \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \,. \end{cases}$. Prove that the Lebesgue decomposition of ν with respect to μ, $\nu = \nu_{ac} + \nu_s$, is given by $\nu_{ac} \equiv 0$ and $\nu_s = \nu$.

I.11 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \overline{\mathbb{R}}$ be a measurable function. Suppose at least one of $f^+ := f \vee 0$ or $f^- := (-f) \vee 0$ is integrable, and let $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ denote the extended real-valued function on $\mathcal{M}$ defined by

$$\nu(E) := \int_E f \, d\mu \quad \forall E \in \mathcal{M} \,.$$

Prove that ν is a signed measure.

I.12 Let (X, d) be a locally compact metric space. Let ν and μ be Radon measures on X . Suppose that
(i) $w \in L^1(X, \mu)$, $w \geqslant 0$ μ- a.e. on X;
(ii) w is continuous at x_0 and $\mu(B(x_0, r)) > 0 \; \forall r > 0$;
(iii) $\nu(A) = \int_A w \, d\mu$ for each $A \in \mathcal{M}$.
Then

$$\exists \lim_{r \to 0} \frac{\nu(B(x_0, r))}{\mu(B(x_0, r))} = w(x_0) \,.$$

I.13 If $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing prove that there is a right-continuous, nondecreasing function $g : \mathbb{R} \to \mathbb{R}$ such that
(i) $g(x) = f(x)$ except for a countable number of $x \in \mathbb{R}$.
(ii) If $g'(x)$ exists at x, so does $f'(x)$ and $g'(x) = f'(x)$.
(**Hint:** Define $g(x) := \lim_{y \to x^+} f(y) \in \mathbb{R}$ if $x \in \mathbb{R}$.
(i) Prove that g is nondecreasing, right-continuous and keep in mind exercise I.7.

(ii) Prove that: the continuity points of f and g agree, at each point x of continuity $f(x) = g(x)$ and the following inequality holds

$$g\left(x + \left(1 - \varepsilon \frac{h}{|h|}\right) h\right) - g(x) \leqslant f(x+h) - f(x) \leqslant g(x+h) - g(x)$$

for each $0 < \varepsilon < 1$, $h \neq 0$.)

I.14* (Cantor set, see also [GZ, section 4.4]) Let $C \subset \mathbb{R}$ denote the set, called Cantor (ternary) set, constructed in stages as follows.

At the first step let $I_{1,1} := (\frac{1}{3}, \frac{2}{3})$. Thus $I_{1,1}$ is the open middle third of the interval $[0, 1]$. Also denote $C_1 := [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. The second step involves performing the first step on each of the two remaining intervals of C_1. That is, we produce two open intervals $I_{2,1}$ and $I_{2,2}$, each being the open middle third of one of the two intervals comprising C_1. Again denote $C_2 := C_1 \setminus (I_{2,1} \cup I_{2,2})$. At the i^{th} step we produce 2^{i-1} open intervals, $I_{i,1}, \dots, I_{i,2^{i-1}}$, each of lenght $(\frac{1}{3})^i$. Denote $C_i := C_{i-1} \setminus (I_{i,1} \cup \dots I_{i,2^{i-1}})$ and define $C := \cap_{i=1}^{\infty} C_i$. Then
(i) C is a compact set and $\mathcal{L}^1(C) = 0$;
(ii) C is *nowhere dense* (that is, C does not contain any (non empty) open set of $\mathbb{R}$);
(iii) C has the cardinality of the continuum (that is, $\text{card}(C) = c = \text{card}(\mathbb{R})$).
(**Hint:** (iii) Prove that the numbers $x \in C$ agree with $x \in [0, 1]$ whose ternary expansion admits the digits either 0 or 2. Here we adopt the convenction to represent each numver $x \in [0, 1]$ by means of a ternary expansion which has the smallest number of digits equal to 1. Therefore C has the cardinality of the set $2^{\mathbb{N}}$.)

I.15* (Cantor-Lebesgue-Vitali function, see also [GZ, section 5.6]) Let C_i $(i \in \mathbb{N})$ denote the sets defined in the exercise I.12. Let $J_{i,1}, \dots J_{i,2^i-1}$ denote the open intervals such that $[0, 1] \setminus C_i = \cup_{h=1}^{2^i-1} J_{i,h}$ and order them in the obvious way from left to right. Given $i \in \mathbb{N}$, let $f_i : [0, 1] \to [0, 1]$ be the continuous function defined as follows: $f_i(0) = 0$, $f_i(1) = 1$, $f_i(x) = \frac{h}{2^i}$ if $x \in J_{i,h}$, and f_i linearly on C_i. Prove that the sequence of functions $(f_i)_i$ converges uniformly to a function $f : [0, 1] \to [0, 1]$ (called *Cantor-Lebesgue-Vitali function*) verifying the following properties:
(i) f is continuous, nondecreasing and $f(0) = 0$, $f(1) = 1$.
(ii) $\exists f'(x) = 0$ for each $x \in [0, 1] \setminus C$ where C denotes the Cantor set defined in the exercise I.14. Moreover $f' = 0$ a.e. on $[0, 1]$.
(iii) $f(C) = [0, 1]$.
(iv) Extend $f : \mathbb{R} \to \mathbb{R}$ by defining $f(x) = 1$ if $x \geqslant 1$ and $f(x) = 0$ if $x \leqslant 0$ and denote λ_f the Lebesgue- Stieltjes measure induced by f on $\mathbb{R}$. Prove that λ_f and $\mathcal{L}^1$ are (mutually) singular.
Remark: From exercise I.15 (ii) and (iv) it follows that the equality $f(1) - f(0) = \int_0^1 f'(x)\, dx$ does not hold, even though the derivative of f vanishes a.e. in $[0, 1]$! Also notice that, from the assertion (iii), f carries a set of measure 0 onto the interval $[0, 1]$.
(**Hint:** (i): Prove that the sequence $(f_i)_i$ converges uniformly by means of the estimate

$$|f_i(x) - f_{i+1}(x)| \leqslant \frac{1}{2^i} \quad \forall x \in [0, 1],\, i \in \mathbb{N}\,.$$

(ii) The assertion follows noticing that $f(x) = \frac{h}{2^i}$ if $x \in J_{i,h}$.

(iii) The continuity of f implies that $f([0,1]) = [0,1]$ and $f(C)$ is compact. On the other hand, by construction, $f([0,1] \setminus C)$ is denumerable, thus $f(C) = [0,1]$.
(iv) The assertion follows noticing that $\lambda_f(\mathbb{R} \setminus C) = \mathcal{L}^1(C) = 0$.)

I.16 Let $f : [0, \frac{1}{\pi}] \to \mathbb{R}$ be the function

$$f(x) := \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } 0 < x \leqslant \dfrac{1}{\pi} \\ 0 \quad \text{if } x = 0\,. \end{cases}$$

Prove that f has unbounded variation.

I.17 Let $f : [a,b] \to \mathbb{R}$ and let $V_f(a;b)$ denote the total variation of f on $[a,b]$. Prove that
(i) if $g : [a,b] \to \mathbb{R}$, then $V_{f+g}(a;b) \leqslant V_f(a;b) + V_g(a;b)$.
(ii) $V_{cf}(a;b) = |c|\, V_f(a;b)$ for each $c \in \mathbb{R}$.
(iii) $V_f(a;b) = V_f(a;c) + V_f(c;b)$ for each $c \in [a,b]$.
(iv) if f is of bounded variation, then the function $g : [a,b] \to \mathbb{R}$, $g(x) := V_f(a;x)$ if $x \in [a,b]$ is nondecreasing.

I.18 (Example of a continuous function nowhere differentiable) Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = |x|$ if $x \in [-1,1]$, extented with period 2 outside $[-1,1]$ (that is, $g(x+2) = g(x)$ for each $x \in \mathbb{R}$).
(i) Prove that the series

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g(4^n\, x)$$

converges for each $x \in \mathbb{R}$.
(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be the sum of the previous series, that is

$$f(x) := \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g(4^n\, x) \quad x \in \mathbb{R},.$$

Prove that f is continuous.
(iii)* Prove that f is nowhere differentiable on $\mathbb{R}$.
(iv) Given $a,\, b \in \mathbb{R}$ with $a < b$, is $f : [a,b] \to \mathbb{R}$ of bounded variation?
(**Hint:** (i), (ii) and (iii): see [R1], Theorem 7.18.)

I.19 Let $f : [a,b] \to \mathbb{R}$ be a Lipschitz function, that is, by definition, $\exists\, L > 0$ such that $|f(x) - f(y)| \leqslant L\,|x-y|$ for each $x,\, y \in [a,b]$. Prove that $f \in AC([a,b])$.
Remark: From the previous exercise it follows that a Lipschitz function $f : [a,b] \to \mathbb{R}$ is differentiable a.e. in $[a,b]$ and it satisfies $f(x) - f(a) = \int_a^x f'(t)\, dt$ for each $x \in [a,b]$.
(**Hint:** Use the definition of absolutely continuous function.)

I.20 Let $f \in AC([a,b])$ and let $g : [a,b] \to \mathbb{R}$ be the function $g(x) := V_f(a;x)$ if $x \in [a,b]$, where $V_f(a;x)$ denotes the total variation of f on the interval $[a,x]$. Prove that $g \in AC([a,b])$.
(**Hint:** Use exercise I. 17 (iii) and the definition of absolutely continuous function.)

I.21 Let $f,\, g \in AC([a,b])$. Prove that:

(i) the product function $f\,g \in AC([a,b])$;
(ii) (integration by parts formula)

$$\int_a^b f'(x)\,g(x)\,dx = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f(x)\,g'(x)\,dx\,.$$

(**Hint:** (i) Use the definition of absolutely continuous function. (ii) From the assertion (i), prove that $\exists\,(f\,g)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$ for a.e. $x \in [a,b]$. Then it follows.....)

I.22 Give an example of a function $f : [0,1] \to \mathbb{R}$ that is differentiable everywhere but it is not absolutely continuous.
(**Hint:** Modify the function in exercise I.16)

II. Main spaces of functions and results on Banach and Hilbert spaces.

II.1 Prove that $C^0([a,b])$ is a vector space (with respect to $\mathbb{R}$) with infinite dimension.

II.2 Let $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \dots)$ be a sequence of continuous functions. Suppose there exists a functon $f : A \to \mathbb{R}$ such that $f_h \to f$ uniformly on A. Then f is continuous.

II.3 Prove that $(\mathbf{C}^0([a,b], ||\cdot||_\infty)$ is a Banach space, but it is not a Hilbert space.
(**Hint:** Prove that the norm $\|\cdot\|_\infty$ does not satisfy the *parellologram identity*

$$\|f - g\|_\infty^2 + \|f + g\|_\infty^2 = 2\left(\|f\|_\infty^2 + \|g\|_\infty^2\right) \quad \forall\, f, g \in \mathbf{C}^0([a,b])\,.$$

For instance, consider $[a,b] = [-1,1]$, $f(x) = 1 - |x|$ and $g(x) = 1 - f(x)$.)

II.4 Show that $(C^1(\bar{\Omega}), ||\cdot||_{C^1})$ is a Banach space, where $||u||_{C^1} := \sum_{|\alpha|\leqslant 1} ||D^\alpha u||_\infty$. and $\Omega \subset \mathbb{R}^n$ is a bounded open set.

II.5 Let $M > 0$ be a given constant and let $\mathcal{F} = \{f \in C^1([a,b] : ||f||_{C^1} \leqslant M\}$. Prove that
(i) $\mathcal{F}$ is a relatively compact set of $(C^0([a,b]), ||\cdot||_\infty)$;
(ii) $\mathcal{F}$ is not a compact set of $(C^1([a,b], ||\cdot||_{C^1})$.

II.6 Let $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \dots)$ be a sequence of continuous functions and let $(x_h)_h \subset A$. Suppose that
(i) there exists $f : A \to \mathbb{R}$ such that $f_i \to f$ uniformly on A;
(ii) there exists $x \in A$ such that $x_h \to x$.

Then there exists $\lim_{i\to\infty} f_i(x_i) = f(x)$.

Does the assertion still hold if in the assumption (i), in place of the uniform convergence, we assume the pointwise one?

II.7 Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be normed vector spaces. Endow $E \times F$ with one of the following norms:

$||(x,y)||_{E\times F} = ||x||_E + ||y||_F$
$||(x,y)||_{E\times F} = \sqrt{||x||_E{}^2 + ||y||_F{}^2}$
$||(x,y)||_{E\times F} = \max\{||x||_E, ||y||_F\}$.
(i) Show that $||(x,y)||_{E\times F}$ is actually a norm.
(ii) Show that, if $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ are Banach spaces, then $(E\times F, ||(x,y)||_{E\times F})$ is still a Banach space.
(iii) Show that, if $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ are separable, then $(E\times F, ||(x,y)||_{E\times F})$ is still separable.

II.8 Let $(f_i)_i \subset C^1([a,b])$. Suppose that
(i) $(f_i)_i$ e $(f_i')_i$ are bounded sequences in $(C^0([a,b]), ||\cdot||_\infty)$;
(ii) $(f_i)_i$ e $(f_i')_i$ are equicontinuous sequences on $[a,b]$.
Then $(f_i)_i$ is relatively compact sequence in $(C^1([a,b]), ||\cdot||_{C^1})$, that is, there exists a subsequence of $(f_i)_i$ converging to a function $f \in C^1([a,b])$ with respect to the norm $||\cdot||_{C^1}$.

II.9 Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Prove that
(i) f is uniformly continuous on A;
(ii) there exists a unique Lipschitz function $\bar f : \bar A \to \mathbb{R}$ (with same Lipschitz constant of f) such that $\bar f = f$ on A.

II.10 (i) Let $f \in \mathbf{C}^1([a,b])$. Prove that $f \in Lip((a,b))$ and $||f||_{\mathbf{C}^1} = ||f||_{Lip}$.
(ii) $\mathbf{C}^1([a,b])$ is a closed set of $(Lip((a,b)), ||\cdot||_{Lip})$.

II.11 Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Suppose there exist two constants $L > 0$ and $\alpha > 1$ such that

$$|f(x) - f(y)| \leqslant L\,|x-y|^\alpha \quad \forall x, y \in \Omega\,.$$

Then f is constant on Ω, that is, there exists $c \in \mathbb{R}$ such that $f(x) = c$ for each $x \in \Omega$.

II.12 Given $f \in \mathbf{C}^0([a,b]$, let $||f||_{L^2} := \sqrt{\int_a^b f(x)^2 dx}$. Prove that:
(i) $(\mathbf{C}^0([a,b], ||\cdot||_{L^2})$ is a normed vector space;
(ii) $(\mathbf{C}^0([a,b], ||\cdot||_{L^2})$ is not a Banach space.

III.13 Let $1 \leqslant p \leqslant q \leqslant \infty$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded open set. Show that:

$$||f||_{L^p} \leqslant |\Omega|^{(\frac{1}{p}-\frac{1}{q})}\,||f||_{L^q}$$

(**Hint:** Use Hölder's inequality.).

II.14 Let $\mathbb{R}^\infty := \{x : \mathbb{N} \to \mathbb{R} : \exists\, \bar n \text{ t.c. } x(n) = 0\, \forall n > \bar n\}$,
$l^2 := \{x : \mathbb{N} \to \mathbb{R} : \sum_{n=1}^\infty |x(n)|^2 < \infty\}$ and
$||x||_{l^2} := \sqrt{\sum_{n=1}^\infty |x(n)|^2}$.
Prove that
(i) $(\mathbb{R}^\infty, ||\cdot||_{l^2})$ is a normed vector space, but not a Banach space.

(ii) $(l^2, ||\cdot||_{l^2})$ is a Hilbert space;
(iii) $(\mathbb{R}^\infty, ||\cdot||_{l^2})' \equiv l^2$;
(iv) $\mathbb{R}^\infty$ is dense in $(l^2, ||\cdot||_{l^2})$.
(**Hint:** (iii) Prove that for each $f \in (\mathbb{R}^\infty, ||\cdot||_{l^2})'$ there exists $y \in l^2$ such that $f(x) = \sum_{n=1}^\infty y(n)\, x(n)$ for every $x \in l^2$.)

II.15 Let $(E, ||\cdot||)$ be a normed vector space and let $f : E \to \mathbb{R}$ be a linear functional. Show that f is continuous if and only if f is bounded, that is,

$$||f||_{E'} := \sup_{x \in E\setminus\{0\}} \frac{|f(x)|}{||x||} < +\infty \,.$$

II.16 Find a sequence $(f_n)_n \subset L^p([0,1])$, with $1 \leqslant p < \infty$, satisfying:
(i) $f_n \to 0$ in $L^p([0,1])$;
(ii) $\forall x \in [0,1],\ (f_n(x))_n \subset \mathbb{R}$ does not converge to 0.

II.17 Let $(E, ||\cdot||)$ be a normed vector space and let $f : E \to \mathbb{R}$ be a linear functional. Show that, if $\dim E < \infty$, then f is continuous.

II.18 Let $(E, ||\cdot||)$ be a normed vector space and let $E' := \{f : E \to \mathbb{R} : f$ linear and continuous$\}$. Prove that:
(i) E' is a vector space;
(ii) if $||\cdot||_{E'}$ denotes the canonical norm, then $(E', ||\cdot||_{E'})$ is a Banach space.

II.19 Assume that E is a f.d.v.s and $n = \dim_{\mathbb{R}} E$.
(i) If $\mathcal{B} \subset E$ is a basis, then $\mathcal{B}$ generates E, namely $\text{span}_{\mathbb{R}}\mathcal{B} = E$.
(ii) E and $\mathbb{R}^n$ are linearly isomorphic.
(iii) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. Then $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are topologically isomorphic.
Recall that, given $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ normed vector spaces, they are said to be algebraically and topologically isomorph if there exists a linear continuous isomorphism $T : (E, ||\cdot||_E) \to (F, ||\cdot||_F)$, with inverse $T^{-1} : (F, ||\cdot||_F) \to (E, ||\cdot||_E)$ continuous.
(iv) Let $\|\cdot\|$ be a norm on E. Then $(E, \|\cdot\|)$ and $(E', \|\cdot\|_{E'})$ are topologically isomorphic. Recall that, given $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ normed vector spaces, they are said to be linearly and topologically isomorphic if there exists a linear continuous isomorphism $T : (E, ||\cdot||_E) \to (F, ||\cdot||_F)$, with inverse $T^{-1} : (F, ||\cdot||_F) \to (E, ||\cdot||_E)$ continuous.

II.20 Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be normed vector spaces and let

$$T : ((E \times F)', ||\cdot||_{(E\times F)'}) \to (E' \times F', ||\cdot||_{E'\times F'})$$

$$f \to (f(\cdot, 0), f(0, \cdot))\,.$$

Prove that T is a linear and topological isomorphism.

II.21 Let $(E, ||\cdot||_E)$ be a normed vector space and let H denote the hyperplane $H = \{f = \alpha\}$. Show that
H is a closed set $\Leftrightarrow f \in E'$.

II.22 Let $(E, ||\cdot||)$be a normed vector space and $F \subset E$ a subspace.
Show that:
(i) if $\dim_{\mathbb{R}} F < +\infty$, then F is closed;
(ii) all the subspaces of $\mathbb{R}^n$ are closed in $\mathbb{R}^n$;
(iii) $\overline{F}$ is also a subspace.

II.23 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and denote by $\mathrm{spt}_e(f)$ the *essential support* of f, that is $\mathrm{spt}_e(f) := \mathbb{R}^n \setminus A_f$, where

$$A_f := \bigcup_{\omega \in \mathcal{A}_f} \omega \qquad \mathit{and} \qquad \mathcal{A}_f := \{\omega \subset \mathbb{R}^n : \omega \,\text{open}\,,\, f = 0 \quad \text{a.e. on}\, \omega\}\ .$$

Prove that

$$\mathrm{spt}_e(f) = \mathrm{spt}(f)\,,$$

that is,

$$\mathbb{R}^n \setminus A_f = \text{closure}\,\{x \in \mathbb{R}^n :\ f(x) \neq 0\}$$

II.24 Let $\Omega \subset \mathbb{R}^n$ be an open set, let $h : \Omega \to \mathbb{R}$ and $\varrho : \Omega \to [0, +\infty)$ be Lebesgue measurable functions and suppose that $\int_\Omega \varrho\, dx = 1$. Prove that for each $p \in [1, +\infty)$

$$\left(\int_\Omega |h|\, \varrho\, dx\right)^p \leqslant \int_\Omega |h|^p\, \varrho\, dx$$

II.25 Let $u \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, with $1 \leqslant p < q \leqslant \infty$. Prove that $u \in L^r(\mathbb{R}^n)$ for every $r \in [p, q]$ and

$$\|u\|_{L^r} \leqslant \|u\|^{\alpha}_{L^p}\, \|u\|^{1-\alpha}_{L^q}$$

provided that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

II.26 Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be normed vector spaces. Denote

$$\mathcal{L}(E, F) := \{T : E \to F \ \text{ such that } \ T \ \text{ is linear and continuous}\}$$

Show that $(\mathcal{L}(E, F), ||\cdot||_{\mathcal{L}(E,F)})$ is a normed vector space, endowed with the norm

$$||T||_{\mathcal{L}(E,F)} := \sup\{||T(x)||_F : x \in B_E\}\,.$$

II.27 Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be, respectively, a Banach and a normed vector space. Let $(T_n)_n \subset \mathcal{L}(E, F)$. Suppose there exists

$$Tx := \lim_{n\to\infty} T_n x \qquad \forall x \in E$$

Show that:
(i) $\sup_{n\in\mathbb{N}} ||T_n||_{\mathcal{L}(E,F)} < +\infty$;
(ii) $T \in \mathcal{L}(E, F)$;
(iii) $||T||_{\mathcal{L}(E,F)} \leqslant \liminf_{n\to\infty} ||T_n||_{\mathcal{L}(E,F)}$.

(**Hint:** Use the Banach- Steinhaus theorem.)

II.28** Given $(E, ||\cdot||)$ normed vector space with $\dim_{\mathbb{R}} E = \infty$, prove there exists a nonlinear operator $T : E \to E$ continuous, bijective but with inverse $T^{-1} : E \to E$ discontinuous.
(**Hint (proposed by M. Degiovanni):** Denote with $S = \{u \in E : \|u\| = 1\}$ the unit sphere of E. Then S is not compact.
Prove, because of the Tietze extension theorem, that there exists an unbounded function $f : S \to [1, +\infty)$.
Define $T : E \to E$ by $T(u) = \frac{u}{f(u/\|u\|)}$ if $u \neq 0$ and $T(0) = 0$. Prove that T is continuous and bijective.
Prove there exists a non convergent sequence $(u_h)_h \subset S$ with $\lim_{h\to\infty} f(u_h) = +\infty$ such that $v_h := \lim_{h\to\infty} T(u_h) = 0$. Hence the sequence $(u_h = T^{-1}(v_h))_h$ does not converge, even though the sequence $\lim_{h\to\infty} v_h = 0$.)

II.29 Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be Banach spaces. If $T : E \to F$ is a linear operator, denote by $G(T)$ its graph, that is $G(T) := \{(x, Tx) : x \in E\}$.
Show that $G(T)$ is closed if and only if $(E, ||\cdot||_1)$ is a Banach space, where$||x||_1 := ||x||_E + ||Tx||_F$.
(**Hint:** Use the closed graph theorem.)

II.30 Let $E = C^0([a, b]) \times \mathbb{R}^n$, $||v||_E := ||f||_\infty + ||w||_{\mathbb{R}^n}$ if $v = (f, w) \in E$. Show that $(E, ||\cdot||_E)$ is a Banach space.

II.31 Consider the following initial value problem (or Cauchy problem) for the linear ordinary differential equation (ode):

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t) u^{(i)} = f$$

with initial values: $u(t_0) = w_1, \ldots, u^{(n-1)}(t_0) = w_n$, where $a_i \in C^0([a, b])$ $(i = 1, \ldots, n-1)$, $w_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and $t_0 \in [a, b]$ are given. Define the operator $T : E := C^0([a, b]) \times \mathbb{R}^n \to F := C^n([a, b])$, defined by $T(f, w) = u$, where (f, w) are the given data problem and u denotes the (unique) problem solution. Prove that:
(i) T is a linear operator.
(ii) $T : (E, ||\cdot||_E) \to (F, ||\cdot||_F)$ is continuous, where $||\cdot||_E$ denotes the norm defined in exercise II.30 and $||\cdot||_F := ||\cdot||_{C^n}$.
Remark: From exercise II.31 (ii), it follows the continuity of the solutions with respect to the initial data: the so-called *well-posedness* of the Cauchy problem for a linear ode of order n.
(**Hint:** Use the closed graph theorem.)

II.32 Let $E = C^1([a, b])$, $F = C^0([a, b])$, $||\cdot||_E = ||\cdot||_F = ||\cdot||_\infty$. Define the operator $T : C^1([a, b]) \to C^0([a, b])$ $Tu = u'$. Prove that the graph of T is closed in $E \times F$, but T is discontinuos. Why does not the closed graph theorem hold?

II.33 Let $(H, (\cdot,\cdot))$ be a Hilbert space.
(i) Define $T : H \to H'$

$$u \mapsto Tu$$

where

$$< Tu, v >_{H' \times H} := (u, v)_H \,.$$

Show that T is a linear, onto isometry.
(ii) $(H', (\cdot,\cdot)_{H'})$ is a Hilbert space, if

$$(f, g)_{H'} := \; (T^{-1}(f), T^{-1}(g)) \quad \text{if } f, g \in H' \,,$$

and

$$\|f\|_{H'} = \; \sqrt{(f,f)_{H'}}$$

II.34 Let

$$\mathcal{B} := \left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{\cos hx}{\sqrt{\pi}} : h = 1, 2, \cdots \right\} \cup \left\{ \frac{\sin hx}{\sqrt{\pi}} : h = 1, 2, \cdots \right\}.$$

Show that $\mathcal{B}$ is a Hilbert basis of $L^2(-\pi, \pi)$, that is:
(i) $\mathcal{B}$ is an orthonormal system of H;
(ii)* $\operatorname{span} \mathcal{B}$ is dense in $L^2(-\pi, \pi)$.
(**Hint:** (ii): see [R2] section 4.24.)

II.35 Let $(E, ||\cdot||)$ be a normed vector space. Prove that, for fixed $x \in E$, it holds that

$$||x|| = \sup\{< f, x >_{E' \times E} \colon f \in B_{E'}\} = \max\{< f, x >_{E' \times E} \colon f \in B_{E'}\} \,.$$

II.36 Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be Banach spaces, $S \in \mathcal{L}(X, Y)$. let $S' : Y' \to X'$ denote the *adjoint* of S, that is, $g = S'(f)$ is defined by

$$g(x) := \; f(S(x)) \quad \forall x \in X \,,$$

if $f \in Y'$. Prove that:
(i) $S' \in \mathcal{L}(Y', X')$;
(ii) if S is dense, that is, $S(X)$ is dense in Y, then $S' : \, Y' \to X'$is injective.

II.37 Let $l^1 := \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x(n)| < \infty) := ||x||_{l^1}\}$and
$l^\infty := \{x \in \mathbb{R}^{\mathbb{N}} : \sup |x(n)| < \infty)$, equipped with the norm $||x||_{l^\infty} := \sup_{n \in \mathbb{N}} |x(n)|$.
Show that:
(i) $(l^1)' \equiv l^\infty$, that is, $f \in (l^1)' \Leftrightarrow \exists ! y \in l^\infty$ such that $< f, x >= \sum_{n=1}^{\infty} y(n)x(n) \;\forall x \in l^1$.
(ii) $(l^1, ||\cdot||_{l^1})$ is a separable Banach space, but $(l^\infty, ||\cdot||_{l^\infty})$ is not a separable Banach space.

II.38 Let $(E, ||\cdot||_E)$ be a Banach space and let $(x_n)_n \subset B_E$.

Denote $M_0 := \text{span}_{\mathbb{R}}\{x_h : h \in \mathbb{N}.\}$ Show that M_0 is separable.

II.39 Let $(X, \mathcal{M}, \mu)$ be the measure space where $X = \mathbb{N},\ \mathcal{M} = \mathcal{P}(X),\ \mu = \#$ and $\#$ denotes the counting measure on $\mathbb{N}$.
Show that:
(i) $L^p(X, \mu) = l^p := \{f : \mathbb{N} \to \mathbb{R} : \sum_{n=1}^{\infty} |f(n)|^p < \infty\}$ if $1 \leqslant p < \infty$;
(ii) $L^\infty(X, \mu) = l^\infty := \{f : \mathbb{N} \to \mathbb{R} : \sup_n |f(n)| < \infty\}$ if $p = \infty$.

III. Complements on Banach and Hilbert spaces.

III.1 Given $(f_h)_h \subset E'$, $f \in E'$, $(x_h)_h \subset E$ and $x \in E$, show that, if $f_h \to f$ e $x_h \rightharpoonup x$,, it follows that

$$< f_h, x_h > \to < f, x > .$$

III.2 Let $E = C^0([0,1]),\ ||\cdot||_E = ||\cdot||_\infty,\ (f_h)_h \subset E,\ f \in E.$ Prove that:
(i) if $f_h \rightharpoonup f$ in E, then
(a) $\exists M > 0$ such that $|f_h(x)| \leqslant M \forall x \in [0,1],\ \forall h \in \mathbb{N}$;
(b) $f_h(x) \to f(x),\ \forall x \in [0,1]$.
(ii) $f_h(x) := x^h,\ x \in [0,1]$ does not weakly converge in E;
(iii) $(C^0([0,1]), ||\cdot||_\infty)$ is not reflexive.

III.3 Let $E = C^1([0,1]),\ ||\cdot||_E = ||\cdot||_{C^1},\ (f_h)_h \subset E,\ f \in E.$ Prove that:
(i) if $f_h \rightharpoonup f$ in E, then $f_h(x) \to f(x)$ e $f_h'(x) \to f'(x),\ \forall x \in [0,1]$;
(ii) the sequence $f_h(x) := \frac{x^h}{h},\ x \in [0,1]$ does not weakly converge in E;
(iii) $(C^1([0,1]), ||\cdot||_{C^1})$ is not reflexive.

III.4 $(Lip((0,1)), \|\cdot\|_{Lip})$ is not reflexive.
(**Hint:** By contradiction: suppose that $(Lip((0,1)), \|\cdot\|_{Lip})$ is reflexive. From exercise III. 10 (ii) it follows that $(C^1([0,1]), \|\cdot\|_{C^1})$ is reflexive: why? From exercise III.3 it follows an absurd.)

III.5 Let $E = l^\infty := \{x \in \mathbb{R}^{\mathbb{N}} : \sup |x(n)| < \infty)$ equipped with the norm $||x||_{l^\infty} := \sup_{n \in \mathbb{N}} |x(n)|$.
(i) Show that $(l^\infty, ||\cdot||_{l^\infty})$ is a Banach space, but it is not separable.
(ii) Consider $(\delta_n)_n \subset (l^\infty)'$ defined by

$$< \delta_n, x > := x(n)\,.$$

Prove that $||\delta_n||_{(l^\infty)'} = 1\ \forall n \in \mathbb{N}$.
(iii) Show that it does not exist any subsequence of $(\delta_n)_n$ weakly converging in $(l^\infty)'$.
(**Hint:** (i) By contradiction: suppose $\exists\, D = \{\, d_i : i \in \mathbb{N}\} \subset l^\infty$ dense. Define $z \in l^\infty$ in the following manner: $z(h) = 2$ if $|d_h(h)| < 1$ and $z(h) = 0$ if $|d_h(h)| \geqslant 1$. Then $||z - d_i||_{l^\infty} \geqslant 1$ for every $i \in \mathbb{N}$ and therefore....

(iii) Let $(h_k)_k$ be an increasing sequence of positive integers and define $x \in l^\infty$ in the following manner: $x(h) = (-1)^k$ if $h = h_k$ and $x(h) = 0$ otherwise. Then, $< \delta_{h_k}, x >= (-1)^k$ does not converge.)
Remark: From exercise III.5 (iii) it follows that $B_{E'}$ is not sequentially compact with respect to $\sigma(E', E'')$.

III.6 Let $f_h : [0,1] \to \mathbb{R}$ $(h = 1, 2, \dots)$ be the functions defined by $f_h(x) = h$ if $0 \leqslant x \leqslant 1/h$ and $f_h(x) = 0$ otherwise. Prove that it does not exist a subsequence $(f_{h_k})_k$ and a $f \in L^1((0,1))$ such that $f_{h_k} \rightharpoonup f$ in $L^1((0,1))$.
(**Hint:** By contradiction: there exists $(f_{h_k})_k$ and a $f \in L^1((0,1))$ such that $f_{h_k} \rightharpoonup f$ in $L^1((0,1))$. Then, there exists a sequence $(g_i)_i \subset L^\infty(0,1))$ such that

$$\lim_{k\to\infty} \int_0^1 f_{h_k}\, g_i dx = 1 \quad (\, i \,\text{fixed}) \quad \text{and} \lim_{i\to\infty} \int_0^1 f\, g_i dx = 0 \,.$$

Thus, a contradiction follows.)
Remark: From exercise III.6 it follows that $(L^1((0,1)), ||\cdot||_{L^1})$ is not reflexive: why?)

III.7 Let $E = C^0([-1,1])$, $||\cdot||_E = ||\cdot||_\infty$ and let $f : E \to \mathbb{R}$ denote the functional defined by

$$f(u) = \int_{-1}^1 \text{sign}(x) u(x) dx \,,$$

where $\text{sign}(x) := \frac{x}{|x|}$ se $x \in [-1,1] \setminus \{0\}$. Show that:
(i) $f \in E'$;
(ii) $||f||_{E'} = 2$;
(iii)* $\nexists\ max_{B_E} f$, that is, there is no $u \in B_E$ such that $f(u) = 2$.
(**Remark:** Exercise III.7 (iii) implies that E is not reflexive: why?
Furthermore, compare exercise III.7 with exercise II.35)

III.8 Let $(E, ||\cdot||)_E$ and $(F, ||\cdot||)_F$ be reflexive Banach spaces. Show that $(E \times F, ||\cdot||_{E\times F})$ (equipped with any one of the norms introduced in exercise II.7) is still a reflexive Banach space.
(**Hint:** Use exercise II. 20.)

III.9 Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be normed vector spaces and let $T : (E, ||\cdot||_E) \to (F, ||\cdot||_F)$ denote a linear continuous operator. Prove that

$$x_h \rightharpoonup x \quad in \quad E \quad \Rightarrow \quad T(x_h) \rightharpoonup T(x) \quad in \quad F \,.$$

(**Hint:** see [B] Theorem 3.9.)

III.10 Let $(H, (\cdot,\cdot))$ be a Hilbert space and let $(x_h)_h \subset H$, $x \in H$. Prove that:
(i)

$$x_h \rightharpoonup x \iff (x_h, y) \to (x, y) \quad \forall y \in H \,;$$

(ii)

$$x_h \to x \iff x_h \rightharpoonup x \quad \text{and} \quad ||x_h|| \to ||x|| \,.$$

III.11 Let $H = L^2((-\pi,\pi))$, $(\cdot,\cdot) = (\cdot,\cdot)_{L^2}$ and let $f_h(x) := sin(hx)$ if $x \in (-\pi,\pi)$. Prove that:
(i)

$$f_h \rightharpoonup 0\,;$$

(ii) $(f_h)_h$ does not (strongly) converge to 0 in $L^2((-\pi,\pi))$.
(**Hint:** (i) use exercise II.34; (ii) use exercise III.10 (ii).)

III.11.1* Let $p \in [1,\infty]$, $f \in L^p_{\text{loc}}(\mathbb{R})$ a periodic function with period $T > 0$, that is $f(x+T) = f(x)$ for a.e. $x \in \mathbb{R}$, and let $(f_h)_h \subset L^p_{\text{loc}}(\mathbb{R})$ be the sequence defined by

$$f_h(x) := f(hx) \text{ if } x \in \mathbb{R},\ h \in \mathbb{N}\,.$$

Let $\bar{f}$ denote the average of f on $[0,T]$, that is

$$\bar{f} = ⨍_0^T f(y)\,dy := \frac{1}{T}\int_0^T f(y)\,dy\,.$$

Prove that, for each $a,b \in \mathbb{R}$, $a < b$,

(i) $f_h \rightharpoonup \bar{f}$ in $L^p((a,b))$, if $1 < p < \infty$;
(ii) $f_h \overset{*}{\rightharpoonup} \bar{f}$ in $L^\infty((a,b))$, if $p = \infty$.
(**Hint:** WLOG we can assume that $T = 1$, otherwise we can replace $f(x)$ with $f(Tx)$. Let us denote $Y_{h,k} := \left[\frac{k}{h}, \frac{k+1}{h}\right)$ for $k \in \mathbb{Z}$ and $h \in \mathbb{N}$. If $g \in \mathbf{C}^0_c((a,b))$ let $g_h : \mathbb{R} \to \mathbb{R}$ denote the step function defined by $g_h(x) = g(k/h)$ if $x \in Y_{h,k}$. Prove the following steps.
1st step: By the uniform continuity of g on $[a,b]$, $g_h \to g$ uniformly on (a,b). In particular, $g_h \to g$ in $L^q((a,b))$ for each $q \in [1,\infty]$.
2nd step: If $1/h < \text{dist}(\text{spt}(g), \mathbb{R}\setminus(a,b))$, $\text{spt}(g_h) \subset (a,b)$ and, by the periodicity of f,

$$\int_a^b f_h(x)\,g_h(x)\,dx = \bar{f}\int_a^b g_h(x)\,dx$$

3rd step: $(f_h)_h$ is bounded in $L^p((a,b))$.
4th step: By Hölder inequality, for each $g \in \mathbf{C}^0_c((a,b))$, it follows that, if $q = p'$ when $1 < p < \infty$ and $q = 1$ when $p = 1$,

$$\begin{aligned}
&\left|\int_a^b \left(f_h(x) - \bar{f}\right) g(x)\,dx\right| \\
&= \left|\int_a^b \left(f_h(x) - \bar{f}\right)(g(x) - g_h(x))\,dx\right| \\
&+ \left|\int_a^b \left(f_h(x) - \bar{f}\right) g_h(x)\,dx\right| \\
&\leqslant \|f_h - \bar{f}\|_{L^p}\|g_h - g\|_{L^q}\,.
\end{aligned}$$

This implies, from the previous steps, that

$$\lim_{h\to\infty}\int_a^b \left(f_h(x) - \bar{f}\right) g(x)\,dx = 0 \text{ for each } g \in \mathbf{C}^0_c((a,b))\,.$$

By the the density of $\mathbf{C}_c^0((a,b))$ in $L^q((a,b))$, (i) and (ii) follow.)

III.12 Let $E = L^1((0,1))$, $||\cdot|| = ||\cdot||_{L^1}$ and let

$$C := \left\{ u \in L^1((0,1)) : \, u(x) \geqslant 0 \text{ a.e. } x \in (0,1)\,, \int_0^1 x\, u(x) dx \geqslant 1 \right\} .$$

Show that:
(i) C is a nonempty, closed and convex set of E;
(ii) $d(0,C) := \inf \{||u|| : u \in C\} = 1$;
(iii)* there is no $u \in C$ such that $||u|| = d(0,C) = 1$.

III.13 Let $E = \mathbb{R}^n$ and $|\cdot|$ denote the Euclidean norm. Prove that:
(i) $a : E \times E \to \mathbb{R}$ is a bilinear map if and only if there exists a unique real matrix $n \times n$ $A = [a_{ij}]_{ij}$ such that

$$a(u,v) = (Au, v) \quad \forall u, v \in \mathbb{R}^n ,$$

where $(\cdot,\cdot)$ denotes the canonical scalar product on $\mathbb{R}^n$.
(ii) If $a : E \times E \to \mathbb{R}$ is a bilinear map, it is also continuous.
(iii) Let A be a real $n \times n$ matrix such that $(Au, u) > 0 \quad \forall u \in \mathbb{R}^n \setminus \{0\}$. Then A is invertible.
(iv) The inverse implication of the statement (iii) does not hold.
(**Hint:** (i) It follows from well known results of linear algebra. (ii) It follws by recalling the following property : if $\dim E < \infty$ and $f : E \to \mathbb{R}$ is linear, then f is also continuous. (iii) Prove that the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ $T(u) := Au$ is both injective and surjective. (iv) Find a counterexample through a 2×2 matrix.)

III.14 Let $(H, (\cdot,\cdot))$ be a Hilbert space and let $A : H \to H$ denote a linear continuous operator such that there exists a constant $\alpha > 0$ for which

$$(Au, u) \geqslant \alpha\, ||u||^2 \quad \forall u \in H .$$

Then A is invertible and $||A^{-1}||_{\mathcal{L}(H,H)} \leqslant \frac{1}{\alpha}$.
(**Hint:** Prove that $R(A) := \{Au : \, u \in H\}$, the range of A in H, is closed and dense in H, whence $R(A) = H$.)

IV. An introduction to the Sobolev space and an application to Poisson's equation.

Exercise IV.1 (i) If $f \in \mathbf{C}^2(\overline{\Omega})$, denote

$$\|f\|_{\mathbf{C}^2} := \sum_{|\alpha| \leqslant 2} \|D^\alpha f\|_{\infty,\Omega} .$$

Then $(\mathbf{C}^2(\overline{\Omega}), \|\cdot\|_{\mathbf{C}^2})$ is a B.s.
(ii) $(\mathbf{C}^2(\overline{\Omega}), \|\cdot\|_{\mathbf{C}^2})$ is not a reflexive B.s.

(**Hint:** (ii) Let $n = 1$, $\Omega = (0,1)$ and $f_h(x) = \dfrac{x^{h+1}}{(h+1)\,h}$. Prove that $(f_h)_h$ is a bounded sequence in $(\mathbf{C}^2(\overline{\Omega}), \|\cdot\|_{\mathbf{C}^2})$ which does not admit any weakly convergent sequence (compare with Exercise III.2).

Exercise IV.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let

$$\mathcal{A} := \left\{ v \in \mathbf{C}^2(\overline{\Omega}) : v = 0 \text{ on } \partial\Omega \right\}, \quad (u,v)_{\mathcal{A}} := \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^n}\, dx \quad \forall\, u, v \in \mathcal{A}.$$

Prove that :
(i) $(\cdot,\cdot)_{\mathcal{A}}$ is a scalar product on $\mathcal{A}$;
(ii) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a n.v.s, but not a B.s. with $\|u\|_{\mathcal{A}} := \sqrt{(u,u)_{\mathcal{A}}}$.

Exercise IV.3 (Heavised function) Prove that the function $H : \mathbb{R} \to \mathbb{R}$, $H(x) := \chi_{[0,+\infty)}(x)$ does not admit weak derivative in $L^1_{loc}(\mathbb{R})$.
(**Hint:** Notice that, for each $\varphi \in \mathbf{C}^1_c(\mathbb{R})$

$$\int_{\mathbb{R}} \varphi'\, dx = -\,\varphi(0) = -\int_{\mathbb{R}} \varphi\, d\delta_0 .$$

Prove there is no function $v \in L^1_{loc}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \varphi\, d\delta_0 = \int_{\mathbb{R}} v\,\varphi\, dx \quad \forall \varphi \in \mathbf{C}^1_c(\mathbb{R})\,.)$$

Exercise IV.4* (Characterization of $H^1_0((a,b))$) Let a, $b \in \mathbb{R}$ with $a < b$. Prove that

$$H^1_0((a,b)) = \left\{ u \in AC([a,b]) : u' \in L^2(a,b),\, u(a) = u(b) = 0 \right\}$$

that is, by definition, given $u \in L^2(a,b)$, then $u \in H^1_0((a,b))$ iff there exists $\tilde{u} \in AC([a,b])$ such that $u = \tilde{u}$ a.e. on (a,b), $\tilde{u}' \in L^2(a,b)$ and $\tilde{u}(a) = \tilde{u}(b) = 0$.
(**Hint :** The inclusion $H^1_0((a,b)) \subset \{u \in AC([a,b]) : u' \in L^2(a,b),\, u(a) = u(b) = 0\}$ follows by the fundamental theorem of calculus. Indeed, if

$$(u_h)_h \subset \mathcal{A} := \left\{ u \in C^2([a,b]) : u(a) = u(b) = 0 \right\}, u_h \to u \text{ and } u_h' \to v \text{ in } L^2(a,b)\,,$$

then it follows that

$$u(x) = \int_a^x v(t)\, dt \quad \forall x \in [a,b] \quad \text{and} \quad u(b) = \int_a^b v(t)\, dt = 0\,.$$

Thus, from the fundamental theorem of calculus, $u \in AC([a,b]$, $u' = v$ a.e. in (a,b). The reverse inclusion can be proved by arguing as follows. Let $u \in AC([a,b])$, with $u' \in L^2(a,b)$ and $u(a) = u(b) = 0$. Notice that, by fundamental theorem of calculus,

$$u(x) = \int_a^x u'(t)\, dt \quad \forall x \in [a,b] \quad \text{and} \quad u(b) = \int_a^b u'(t)\, dt = 0\,.$$

Let $(\varrho_h)_h$ be a sequence of mollifiers in $\mathbb{R}$ and let $\overline{u'} : \mathbb{R} \to \mathbb{R}$ denote the function defined by $\overline{u'} = u'$ in (a,b) and $\overline{u'} \equiv 0$ otherwise. Let $v_h, u_h : \mathbb{R} \to \mathbb{R}$ be the functions defined respectively by

$$v_h(x) := (\varrho_h * \overline{u'})(x) \quad \text{and} \quad u_h(x) := \int_a^x v_h(t)\, dt \text{ if } x \in \mathbb{R}\,.$$

Prove that

$$(u_h)_h \subset \mathcal{A},\ u_h \to u \text{ and } u_h' \to u' \text{ in } L^2(a,b)\,.)$$

Esercizio IV.5 Let $n = 1$, $\Omega = (a,b)$ with $-\infty < a < b < +\infty$, $f \in L^2(\Omega)$ and $\lambda \in \mathbb{R}$ be given. Consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(v) := \frac{1}{2}\int_\Omega (v'^{\,2} + \lambda\, v^2)\, dx - \int_\Omega f\, v\, dx \quad v \in H_0^1(\Omega)$$

and the boundary value problem

$$(P_0) \qquad \begin{cases} -u'' + \lambda\, u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases},$$

Prove that:

(i)(Dirichlet's principle) for each $\lambda \geqslant 0$, there exists a unique $u \in H_0^1(\Omega)$ solution of the problem

$$\text{(DP)} \qquad I(u) = \min_{H_0^1(\Omega)} I\,,$$

and u satisfies the (weak) Euler-Lagrange equation

$$\text{(EL)} \qquad \int_\Omega (u'\, v' + \lambda\, u\, v)\; dx = \int_\Omega f\, v\, dx \quad \forall v \in H_0^1(\Omega)\,.$$

Viceversa, if $u \in H_0^1(\Omega)$ satisfies (EL), then u is a solution of (DP). A function $u \in H_0^1(\Omega)$ satisfying (EL) is said to be a *weak solution* of problem (P_0).

(ii) (Regularity of the weak solutions) If $f \in C^0(\overline{\Omega})$ and $u \in H_0^1(\Omega)$ is a weak solution of (P_0), then u is a classical solution of (P_0), that is, $u \in C^2(\overline{\Omega})$ and u satisfies (P_0) in a pointwise sense.

(iii) If $u \in C^2(\overline{\Omega})$ is a classical solution of (P_0), then u is a weak solution of (P_0).

(iv) If $\Omega = (0,\pi)$, $\lambda = -1$ and $f \equiv 0$, then the functions $u_1(x) \equiv 0$ and $u_2(x) \equiv \sin x$ are weak solutions of the problem (P_0).

(v) If $\Omega = (0,\pi)$, $\lambda = -1$ and $f \equiv 1$, then there are no weak solutions of the problem (P_0).

Remark: From Exercise IV.5 (iv) and (v) it follows that neither the uniqueness and existence of weak solutions for problem (P_0) can be taken for granted whenever $\lambda < 0$.

Results that will be the content of the final written test. The proofs are also requested.

- 1. Radon-Nikodym's theorem: let ν and μ be two measures on $(X, \mathcal{M})$. Suppose that ν and μ are σ-finite and $\nu << \mu$. Then there exists a measurable function $w : X \to [0, \infty]$, called the Radon-Nikodym derivative of ν with respect to μ and denoted by $w = \dfrac{d\nu}{d\mu}$, such that $\nu = \mu_w$ on $\mathcal{M}$, that is, $\nu(E) = \mu_w(E) := \int_E w\, d\mu \quad \forall E \in \mathcal{M}$.
- 2. Lebesgue's decomposition theorem: let ν and μ be σ-finite measures on a measure space $(X, \mathcal{M})$. Then there is a decomposition of ν such that $\nu = \nu_{ac} + \nu_s$ with $\nu_{ac} << \mu$ and ν_s and μ mutually singular. The decomposition is unique.
- 3. Differentiation of a measure with respect to a regular differentiation basis: let ν be a Radon measure on $\mathbb{R}^n$, then there exists $\lim_{h\to\infty} \dfrac{\nu(E_h(x))}{|E_h(x)|} = \dfrac{d\nu_{ac}}{d\mathcal{L}^n}(x)$ a.e. $x \in \mathbb{R}^n$, whenever $(E_h(x))_h$ is a regular differentiation basis of $\mathcal{L}^n$ at x.
- 4. Lebesgue's differentiation theorem for monotone functions: let $f : [a, b] \to \mathbb{R}$ be non decreasing. Then (i) there exists $f'(x)$ for a.e. $x \in [a, b]$ and (ii) $\int_a^b f'(x)\, dx \leqslant f(b) - f(a)$.
- 5. Jordan's decomposition theorem: Let $f : [a, b] \to \mathbb{R}$. Then the following are equivalent: (i) $f \in BV([a, b])$; (ii) there exist $g, h : [a, b] \to \mathbb{R}$ nondecreasing such that $f = g - h$.
- 6. Fundamental theorem of calculus: let $f : [a, b] \to \mathbb{R}$, then $f \in AC([a, b])$ iff (i) f is differentiable a.e. in $[a, b]$, (ii) f' is integrable in $[a, b]$ and (iii) $f(x) - f(a) = \int_a^x f'(t)\, dt \quad \forall x \in [a, b]$.
- 7. Approximation by continuous functions in L^p: let $\Omega \subset \mathbb{R}^n$ be an open set, then $C^0_c(\Omega)$ is dense in $(L^p(\Omega), \|\cdot\|_{L^p})$, provided that $1 \leqslant p < \infty$.
- 8. Approximation by convolution in $L^p(\mathbb{R}^n)$: let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(\varrho_h)_h$ be a squence of mollifiers. Then (i) $f * \varrho_h \in C^\infty(\mathbb{R}^n)$ for each $h \in \mathbb{N}$; (ii)$\|f * \varrho_h\|_{L^p(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}$ for each $h \in \mathbb{N}$, $f \in L^p(\mathbb{R}^n)$, for every $p \in [1, \infty]$; (iii) $\mathrm{spt}(f * \varrho_h) \subset \mathrm{spt_e}(f) + \overline{B(0, 1/h)}$ for each $h \in \mathbb{N}$ and (iv) if $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant \infty$, then $f * \varrho_h \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for each $h \in \mathbb{N}$, and $f * \varrho_h \to f$ as $h \to \infty$, in $L^p(\mathbb{R}^n)$, provided $1 \leqslant p < \infty$.
- 9. Fundamental lemma of the calculus of variations: let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in L^1_{loc}(\Omega)$. Assume that $\int_\Omega f\, \varphi\, dx = 0 \quad \forall \varphi \in C^\infty_c(\Omega)$. Then $f = 0$ a.e. in Ω.
- 10. Approximation by C^∞ functions in $L^p(\Omega)$: let $\Omega \subset \mathbb{R}^n$ be an open set. Then $C^\infty_c(\Omega)$ is dense in $(L^p(\Omega), \|\cdot\|_{L^p})$, provided that $1 \leqslant p < \infty$.
- 11. Weak sequantial compactness of the unit closed ball: B_E is sequentially compact with respect to the topology $\sigma(E, E')$, provided that $(E, \|\cdot\|_E)$ is a reflexive normed vector space
- 12. $(L^p(\Omega), ||\cdot||_{L^p})$ is reflexive if $1 < p < \infty$, but $(L^p(\Omega), ||\cdot||_{L^p})$ is not reflexive if $p = 1, \infty$.

- 13. Generalized Weierstrass theorem in reflexive spaces: let $(E, \|\cdot\|)$ be a reflexive normed space and let $\varphi : A \subset E \to (-\infty, +\infty]$. Suppose that (i) A is closed and φ is convex; (ii) A is bounded or A is unbounded but there exists $\lim_{x \in A, \|x\| \to +\infty} \varphi(x) = +\infty$; (iii) φ is semicontinuous (with respect to τ_s). Then there exists $\min_A \varphi$.
- 14. Dirichlet's principle: let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\Gamma = \partial\Omega$ of class $\mathbf{C}^1$, let $f \in \mathbf{C}^0(\overline{\Omega})$, $g \in \mathbf{C}^0(\Gamma)$ and $\mathcal{A} := \left\{v \in \mathbf{C}^2(\overline{\Omega}) : v = g \text{ on } \Gamma\right\}$. Assume $u \in \mathcal{A}$ solves the boundary-value problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma \end{cases} .$$

Then

$$\text{(DP)} \qquad I(u) = \min_{v \in \mathcal{A}} I(v) .$$

Viceversa, if $u \in \mathcal{A}$ satisfies (DP), then u solves the boundary-value problem (P).
- 15. Dirichlet's principle in the Sobolev space: let $f \in L^2(\Omega)$ and consider the Dirichlet energy functional $I : H_0^1(\Omega) \to \mathbb{R}$, $\quad I(v) := \frac{1}{2}\int_\Omega |Dv|^2\,dx - \int_\Omega f\,v\,dx \quad v \in H_0^1(\Omega)$. Then there exists a unique $u \in H_0^1(\Omega)$ such that

$$\text{(DP)} \qquad I(u) = \min_{H_0^1(\Omega)} I .$$

Moreover u is characterized by the following property: $u \in H_0^1(\Omega)$ is the unique solution of the Euler-Lagrange equation, in weak form,

$$\text{(EL)} \qquad \int_\Omega (Du, Dv)_{\mathbb{R}^n}\,dx = \int_\Omega f\,v\,dx \quad \forall\, v \in H_0^1(\Omega) .$$

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Final examination procedure : the final examination will be a written test containing three exercises extracted from the attached list and a proof of a result chosen from a results also extracted from the list above. An interview will follow.

INFORMATION ABOUT SOME QUOTED MATHEMATICIANS

Biographical and scientific information more detailed may find at the website http:///www-history.mcs.st-andrews.ac.uk/

- ALAOGLU Leonidas (1914, Red Deer, Alberta, Canada- 1981): Alaoglu was a Canadian-American mathematician, most famous for his widely-cited result called Alaoglu's theorem on the weak-star compactness of the closed unit ball in the dual of a normed space, also known as the Banach-Bourbaki-Alaoglu theorem.
- ASCOLI Giulio (1843, Trieste, Italy - 1896, Milan, Italy): He made contributions to the theory of functions of a real variable and to Fourier series.
- ARZELA' Cesare (1847, Santo Stefano di Magra, La Spezia, Italy - 1912, Santo Stefano di Magra): He conducted a deep research in the field of functional theory.
- BAIRE René-Louis (1874, Paris, France- 1932, Chambéry, France): Baire worked on the theory of functions and the concept of a limit. He is best known for the Baire category theorem, a result he proved in his 1899 thesis.
- BANACH Stefan (1892, Kraków, Austria-Hungary (now Poland) - 1945 in Lvov, (now Ukraine)): Banach founded modern functional analysis and made major contributions to the theory of topological vector spaces. In addition, he contributed to measure theory, integration, and orthogonal series.
- BORSUK Karol (1905, Warsaw,Poland - 1982, Warsaw,Poland) He was mainly interested in topology. His topological and geometric conjectures and themes stimulated research for more than half a century.
- BOREL Emil F.E.J. (1871, Saint Affrique, Aveyron, Midi-Pyrénées, France - 1956, Paris): Borel created the first effective theory of the measure of sets of points, beginning of the modern theory of functions of a real variable.
- BOURBAKI Nicolas: Nicolas Bourbaki is the pseudonym of a group of (mainly) French mathematicians who published an authoritative account of contemporary mathematics.
- CANTOR George F.L.P. (1845, St Petersburg, Russia - 1918, Halle, Germany): Cantor founded the set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers. He also advanced the study of trigonometric series.
- CARATHÉODORY Constantin (1873, Berlin - 1950, Munich): Carathéodory made significant contributions to the calculus of variations, the theory of point set measure, and the theory of functions of a real variable.
- CAUCHY Augustin-Louis (1789, Paris, France - 1857, Sceaux (near Paris), France) Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.
- DIRAC Paul A. (1902, Bristol, England- 1984, Tallahassee, Florida, USA): Dirac is famous as the creator of the complete theoretical formulation of quantum mechanics.
- DIRICHLET Gustav L. (1805, Düren, French Empire (now Germany)- 1859, Göttingen, Hanover (now Germany)) He made valuable contributions to number theory, analysis, and mechanics. In number theory he proved the existence of an infinite

number of primes in any arithmetic series. In mechanics he investigated the equilibrium of systems and potential theory, which led him to the Dirichlet problem concerning harmonic functions with prescribed boundary values.

• EULER Leonhard (1707, Basel, Switzerland -1783, St Petersburg, Russia) He was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.He is also renowed for his work in mechanics, fluid dynamics, optics, and astronomy.

• FATOU Pierre J.L. (1878, Lorient, France - 1929, Pornichet, France): Fatou worked in the fields of complex analytic dynamic and iterative and recursive processes.

• FISHER Ernst (1875, Vienna, Austria - 1954, Cologne, Germany): Ernst Fischer is best known for the Riesz-Fischer theorem in the theory of Lebesgue integration.

• FOURIER Joseph J.B. (1768, Auxerre, Francia - 1830, Parigi): Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

• FRÉCHÉT Maurice (1878, Maligny - 1973, Paris) Fréchét was a French mathematician who made major contributions to the topology of point sets and defined and founded the theory of abstract spaces.In particular, in his thesis he introduced the concept of a metric space, although he did not invent the name 'metric space' which is due to Hausdorff

• FRIEDRICHS Kurt Otto (1901, Kiel, Germany- 1982, New Rochelle, New York, USA) Friedrichs' greatest contribution to applied mathematics was his work on partial differential equations. He also did major research and wrote many books.

• GAUSS Carl F. (1777, Brunswick, Duchy of Brunswick (now Germany) - 1855, Göttingen, Hanover (now Germany)) Gauss worked in a wide variety of fields in both mathematics and physics incuding number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. His work has had an immense influence in many areas.

• GREEN George (1793, Sneinton, Nottingham, England - 1841, Sneinton, Nottingham, England) George Green was an English mathematician best-known for Green's function and Greeen's theorems in potential theory.

• HAHN Hans (1879, Vienna, Austria - 1934, Vienna, Austria): Hahn was an Austrian mathematician who is best remembered for the Hahn-Banach theorem. He also made important contributions to the calculus of variations, developing ideas of Weierstrass.

• HAUSDORFF Felix (1868, Breslau, Germany (now Wroclaw, Poland)- 1942, Bonn, Germany): Hausdorff worked in topology creating a theory of topological and metric spaces. In particular, he introduced the modern notion of metric space. He also worked in set theory and introduced the concept of a partially ordered set.

• HELLY Eduard (1884, Vienna, Austria - 1943, Chicago, Illinois,USA) Helly worked on functional analysis and gave important contributions in this field. For instance, he proved a special form of the Hahn-Banach theorem in 1912, fifteen years before Hahn published essentially the same proof and 20 years before Banach gave his new setting (see [D, Chap. VI, Sect. 2]).

• HILBERT David (1862, Königsberg, Prussia (now Kaliningrad, Russia)-1943, Göttingen, Germany): Hilbert's work in geometry had the greatest influence in that area after Euclid. A systematic study of the axioms of Euclidean geometry led Hilbert to propose 21 such axioms and he analysed their significance. He made contributions in many areas of mathematics and physics.

• HÖLDER Otto L. (1859, Stuttgart, Germany - 1937, Leipzig, Germany): Hölder worked on the convergence of Fourier series and in 1884 he discovered the inequality now named after him. He became interested in group theory through Kronecker and Klein and proved the uniqueness of the factor groups in a composition series.

• KOLMOGOROV Andrey N. (1903, Tambov, Tambov province, Russia - 1987, Moscow) He was a Soviet Russian mathematician, preeminent in the 20th century, who advanced various scientific fields, among them probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity.

• JORDAN Camille M.E. (1838, La Croix-Rousse, Lyon, France - 1922, Paris, France): Jordan was highly regarded by his contemporaries for his work in algebra, group theory and Galois theory. Jordan is best remembered today among analysts and topologists for his proof that a simply closed curve divides a plane into exactly two regions, now called the Jordan curve theorem. He also originated the concept of functions of bounded variation and is known especially for his definition of the length of a curve.

• LAGRANGE Joseph-Louis (1736, Turin, Sardinia-Piedmont (now Italy) - 1813 in Paris, France) Born Giuseppe Lodovico (Luigi) Lagrangia, he was a mathematician and astronomer, lived part of his life in Prussia and part in France, making great contributions to all fields of analysis, to number theory, and to classical and celestial mechanics.

• LEBESGUE Henry L. (1875, Beauvais, Oise, Picardie, France-1941, Paris, France): Lebesgue formulated the theory of measure in 1901 and the following year he gave the definition of the Lebesgue integral that generalises the notion of the Riemann integral.

• LEVI Beppo (1875, Turin, Italy - 1961, Rosarno, Argentina): He studied singularities on algebraic curves and surfaces. Later he proved some foundational results concerning Lebesgue integration.

• LIPSCHITZ Rudolf O.S.(1832, Könisberg, Germany (now Kaliningrad, Russia) -1903, Bonn, Germany) He was a German mathematician and professor at the University of Bonn from 1864. Dirichlet was his teacher. While Lipschitz gave his name to the Lipschitz continuity condition, he worked in a broad range of areas. These included number theory, algebras with involution, mathematical analysis, differential geometry and classical mechanics.

• LUSIN Nikolai N. (1883, Irkutsk, Russia - 1950, Moscow, USSR): Lusin's main contributions are in the area of foundations of mathematics and measure theory. He also made significant contributions to descriptive set topology.

• MINKOWSKI Hermann (1864, Alexotas, Russian Empire (now Kaunas, Lithuania) - 1909, Göttingen, Germany): Minkowski developed a new view of space and time and laid the mathematical foundation of the theory of relativity.

• NIKODYM Otto M. (1887, Zablotow, Galicia, Austria-Hungary (now Ukraine) - 1974, Utica, USA): Nikodym's name is mostly known in measure theory (e. g.

the Radon-Nikodym theorem and derivative, the Nikodym convergence theorem, the Nikodym-Grothendieck boundedness theorem), in functional analysis (the Radon-Nikodym property of a Banach space, the Frechet-Nikodym metric space, a Nikodym set), projections onto convex sets with applications to Dirichlet problem, generalized solutions of differential equations, descriptive set theory and the foundations of quantum mechanics.

• PEANO Giuseppe (1858, Cuneo, Italy - 1932, Turin, Italy): Peano was the founder of symbolic logic and his interests centred on the foundations of mathematics and on the development of a formal logical language. Among his important contributions, let us recall he invented 'space-filling' curves in 1890, these are continuous surjective mappings from [0,1] onto the unit square.

• PRYM Friedrich E. F (1841, Düren, Germany- 1915, Bonn, Gremany) Prym was a German mathematician who introduced Prym varieties and Prym differentials.

• POINCARÉ Jules Henri (1854, Nancy, Meurthe-et-Moselle - 1912, Paris) He was a French mathematician, theoretical physicist, and a philosopher of science. He is often described as a polymath, and in mathematics as The Last Universalist, since he excelled in all fields of the discipline as it existed during his lifetime. As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He is considered to be one of the founders of the field of topology.

• POISSON Siméon D. (1781,Pithiviers, France - 1840, Sceaux near Paris, France) Poisson was a French mathematician, geometer, and physicist.As a scientific worker, his productivity has rarely if ever been equalled. A brief mention of his production includes the application of mathematics to physics that his greatest services to science were performed. Perhaps the most original, and certainly the most permanent in their influence, were his memoirs on the theory of electricity and magnetism, which virtually created a new branch of mathematical physics.

• PRYM Friedrich E. F (1841, Düren, Germany- 1915, Bonn, Gremany) Prym was a German mathematician who introduced Prym varieties and Prym differentials.

• RADON Johann (1887, Tetschen, Bohemia (now Decin, Czech Republic) - 1956, Vienna, Austria): Radon worked on the calculus of variations, differential geometry and measure theory.

• RIEMANN G. F. Bernhard (1826, Breselenz, Hanover (now Germany)- 1866, Selasca, Italy): Riemann's ideas concerning geometry of space had a profound effect on the development of modern theoretical physics. He clarified the notion of integral by defining what we now call the Riemann integral.

• RIESZ Frigyes (Friedrich) (1880, Györ, Austria-Hungary (now Hungary) - 1956, Budapest, Hungary): Riesz was a founder of functional analysis and his work has many important applications in physics.

• RIESZ Marcel (1886, Györ, Austria-Hungary (now Hungary) - 1969, Lund, Sweeden) He was a Hungarian mathematician and moved to Sweden in 1908 and spent the rest of his life there. He was known for work on classical analysis, on fundamental solutions of partial differential equations, on divergent series, Clifford algebras, and number theory. He was the younger brother of the mathematician Frigyes Riesz.

• SCHAUDER Juliusz P. (1899, Lów,Poland - 1943,?) He was a Polish mathematician of Jewish origin, known for his fundamental work in functional analysis, partial differential equation and mathematical physics. Schauder was Jewish, and after the invasion of German troops in Lwów it was impossible for him to continue his work. He was executed by the Gestapo, probably in October 1943.

• SOBOLEV Sergei L. (1908, S.Petersburg - 1989, Moscow): He introduced the notions that are now fundamental for several different areas of mathematics. Sobolev spaces and their embedding theorems are an important subject in functional analysis.

• STEINHAUS Hugo D. (1887, Jaslo, Galicia, Austrian Empire (now Poland) - 1972, Wroclaw, Poland): He did important work on functional analysis. Some of Steinhaus's early work was on trigonometric series. He was the first to give some examples which would lead to marked progress in the subject.

• STIELTJES Thomas J. (1856, Zwolle, Overijssel, The Netherlands - 1894, Toulouse, France): Stieltjes worked on almost all branches of analysis, continued fractions and number theory.

• STONE Marshall H. (1903, New York - 1989, Madras, India) He contributed to real analysis, functional analysis, and the study of Boolean algebras.

• TIETZE Henrich F.F. (1880, Schleinz, Austria - 1964, Munich, Germany): Tietze, famous for the Tietze extension theorem, contributed to the foundations of general topology and developed important work on subdivisions of cell complexes.

• TONELLI Leonida (1885, Gallipoli, Italy - 1946, Pisa, Italy) Tonelli discovered the importance of the semicontinuity in calculus of variations in order to get the existence of minima or maxima for functionals. He also advanced the study of integration theory.

• URYSOHN Pavel S. (1898, Odessa, Ukraine- 1924, Batz-sur-Mer, France): Urysohn is best known for his contributions in the theory of dimension, and for Urysohn's Metrization Theorem and Urysohn's Lemma, both of which are fundamental results in topology.

• VITALI Giuseppe (1875, Ravenna, Italy - 1932, Bologna, Italy): Vitali made significant mathematical discoveries including a theorem on set-covering, the notion and the characterization of an absolutely continuous functions and a criterion for the closure of a system of orthogonal functions.

• VON NEUMANN John (1903, Budapest, Hungary - 1957, Washington D.C., USA): Von Neumann built a solid framework for quantum mechanics. He also worked in game theory, studied what are now called von Neumann Algebras, and was one of the pioneers of computer science.

• WEIERSTRASS Karl (1815, Ostenfelde, Germania -1897, Berlino): Weierstrass is best known for his construction of the theory of complex functions by means of power series. Known as the father of modern analysis, Weierstrass devised tests for the convergence of series and contributed to the theory of periodic functions, functions of real variables, elliptic functions, Abelian functions, converging infinite products, and the calculus of variations.